PROJECTIVE GEOMETRY

Oswald Veblen and John Wesley Youn

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Projective Geometry

VOLUME II

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BLAISDELL PUBLISHING COMPANY

A DIVISION OF GINN AND COMPANY

New York · Toronto · London

A, E; A, E, H_0; A, E, P; or A, E, P, H_0. Among the spaces satisfying A, E, P (the properly projective spaces) may be mentioned the modular spaces, the rational nonmodular space, the real space, and the complex space. Any one of these may be specified categorically by adding the proper assumptions to A, E, P. The passage from the point of view of general projective geometry to that of the particular spaces is made in the first chapter of this volume.

Having fixed attention on any particular space, we have a set of groups of transformations to each of which belongs its geometry. For example, in the complex projective plane we find among others, (1) the group of all-continuous one-to-one reciprocal transformations (analysis situs), (2) the group of birational transformations (algebraic geometry), (3) the projective group, (4) the group of non-Euclidean geometry, (5) a sequence of groups connected with Euclidean geometry (cf. § 54). The groups (2), (3), (4), and (5) all have analogues in the other spaces mentioned in the paragraphs above, and consequently it is desirable to develop the theorems of the corresponding geometries in such a way that the assumptions required for their proofs are put in evidence in each case. This will be found illustrated in the chapters on affine and Euclidean geometry.

The two principles of classification, (*a*) and (*b*), give rise to a double sequence of geometries, most of which are of consequence in present-day mathematics. It is the purpose of this book to give an elementary account of the foundations and interrelations of the more important of these geometries (with the notable exception of (2)). May I venture to suggest the desirability of other books taking account of this logical structure, but dealing with particular types of geometric figures?

The ideal of such books should be not merely to prove every theorem rigorously but to prove it in such a fashion as to show in which spaces it is true and to which geometries it belongs. Some idea of the form which would be assumed by a treatise on conic sections written in this fashion can be obtained from § 83 below. Other subjects for which this type of exposition would be feasible at the present time are quadric surfaces, cubic and quartic curves,

rational curves, configurations, linear line geometry, collineation groups, vector analysis.

Books of this type could take for granted the foundational and coördinating work of such a book as this one, and thus be free to use all the different points of view right from the beginning. On the other hand, a general work like this one could be much abbreviated if there were corresponding treatises on particular geometric figures (for example, conic sections) to which cross references could be made.

OSWALD VEBLEN

BROOKLIN, MAINE
AUGUST, 1917

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PROJECTIVE GEOMETRY

CHAPTER I

FOUNDATIONS

1. Plan of the chapter. In the first volume of this book we have been concerned with general projective geometry, that is to say, with those theorems which are consequences of Assumptions A, E, P. In many cases we also made use of Assumption H_0, but most of the theorems which we proved by the aid of this assumption remain true (though trivial) when this assumption is false. The class of spaces to which the geometry of Vol. I applies is very large, and the set of assumptions used is therefore far from categorical.

The main purpose of geometry is, of course, to serve as a theory of that space in which we envisage ourselves and external nature. This purpose can be accomplished only partially by a geometry based on a set of assumptions which is not categorical. We therefore proceed to add the assumptions which are necessary in order to limit attention to the geometry of reals, the geometry in which the number system is the real number system of analysis.

These assumptions are stated in two ways, the one (§ 3) dependent on the theory of the real number system and the other (§§ 7–13) independent of it. We also state the assumptions (§§ 5, 14, 15, 16) necessary for certain other geometries which are of importance because of their relations to the real geometry and to other branches of mathematics. At the end of the chapter we give a summary of the assumptions for the various projective geometries which we are considering.

"a point is on a line" or "a line is on a point" means that the point belongs to the line (cf. p. 16, Vol. I).

ASSUMPTIONS OF ALIGNMENT:

A 1. *If A and B are distinct points, there is at least one line on both A and B.*

A 2. *If A and B are distinct points, there is not more than one line on both A and B.*

A 3. *If A, B, C are points not all on the same line, and D and E ($D \neq E$) are points such that B, C, D are on a line and C, A, E are on a line, there is a point F such that A, B, F are on a line and also D, E, F are on a line.*

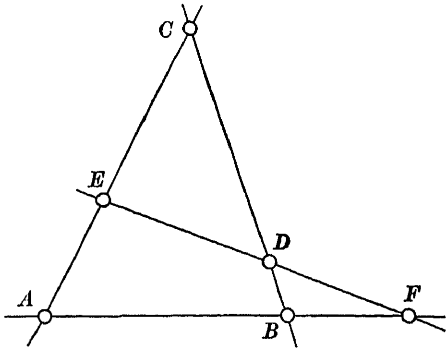


FIG. 1

ASSUMPTIONS OF EXTENSION:

E 0. *There are at least three points on every line.*

E 1. *There exists at least one line.*

E 2. *All points are not on the same line.*

E 3. *All points are not on the same plane.**

E 3′. *If S_3 is a three-space,† every point is on S_3.*

ASSUMPTION OF PROJECTIVITY:

P. *If a projectivity leaves each of three distinct points of a line invariant, it leaves every point of the line invariant.‡*

ASSUMPTION H_0:

H_0. *The diagonal points of a complete quadrangle are noncollinear.§*

As was explained when Assumption P was first introduced, this assumption does not appear in the complete list of assumptions for the geometry of reals, but is replaced by certain other assumptions from which it (as well as H_0) can be derived as a theorem. The list of assumptions for this geometry will consist of Assumptions A, E, and the new assumptions.

3. Assumption K. The most summary way of completing the list of assumptions for the geometry of reals is to introduce the following:

K. *A geometric number system* (Chap. VI, Vol. I) *is isomorphic** *with the real number system of analysis.*

Thus a complete list of assumptions for the geometry of reals is A, E, K.

The use of Assumption K implies a previous knowledge of the real number system.† Its apparent simplicity therefore masks certain real difficulties. What these difficulties are from a geometric point of view will be found on reading §§ 7–13, where K is analyzed into independent statements H, C, R. These sections, however, may be omitted, if desired, on a first reading.

Since a geometric number system in one one-dimensional form is isomorphic with any geometric number system in any one-dimensional form in the same space, it is evident that the principle of duality is valid for all theorems deducible from Assumptions A, E, K.

In order that the results of Vol. I be applicable to the geometry of reals, it must be shown that Assumption P is a logical consequence of Assumptions A, E, K. Since multiplication is commutative in the real number system, this result would follow directly from Theorem 7, Chap. VI, Vol. I. The proof there given is, however, incomplete. It is shown (Theorem 6, loc. cit.) that if P holds, multiplication is commutative; but it is not there proved that if multiplication is commutative, P is satisfied. The needed proof may be made as follows:

THEOREM 1. *Assumption* P *is valid in any space satisfying Assumptions* A *and* E *and such that multiplication is commutative in a geometric number system* (Chap. VI, Vol. I).

Proof. It is obvious that the number systems determined by any two choices of the fundamental points $H_0H_1H_\infty$ are isomorphic (cf. Theorems 1 and 3, Chap. VI, Vol. I), so that we may base our argument on an arbitrary choice of these points. We are assuming that multiplication is commutative, and are to prove that any projectivity Π

* This term is defined in § 52, Vol. I.

† The real number system is to be thought of either as defined in terms which rest ultimately on the positive integers (cf. Pierpont, Theory of Functions of Real Variables, pp. 1–94; or Fine, College Algebra, pp. 1–70) or by means of a set of postulates (cf. E. V. Huntington, Transactions of the American Mathematical Society, Vol. VI (1905), p. 17).

definition, Π is the resultant of a sequence of perspectivities

$$[H] \overset{S_1}{\underset{\wedge}{=}} [P_1] \overset{S_2}{\underset{\wedge}{=}} \cdots \overset{S_n}{\underset{\wedge}{=}} [\Pi(H)]$$

where $[H]$ denotes the points of the given line. By Theorem 5, Chap. III, Vol. I, this chain of perspectivities may be replaced by three perspectivities

$$[H] \overset{S}{\underset{\wedge}{=}} [P] \overset{T}{\underset{\wedge}{=}} [Q] \overset{U}{\underset{\wedge}{=}} [\Pi(H)].$$

Moreover, by Theorem 4, Chap. III, Vol. I, the pencils $[P]$ and $[Q]$ may be chosen so that their respective axes pass through two of the given fixed points of Π. Let us denote these points by H_x and H_y

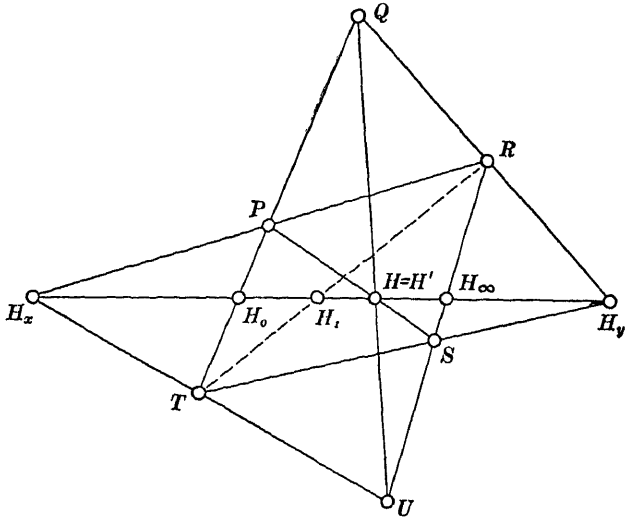


FIG. 2

respectively and let H_∞ be the third fixed point. By another application of Theorem 4 the pencils $[P]$ and $[Q]$ may be chosen so that their common point R is on the line SH_∞ (fig. 2).

Now, since H_∞ is transformed into itself, S, H_∞, and U must be collinear. Since H_x is fixed, T, H_x, and U must be collinear. Since H_y is fixed, S, T, and H_y are collinear. If H is any point of the line H_xH_y, it is transformed by the perspectivity with S as center to a point P of the line H_xR; the perspectivity with T as center transforms P to a point Q of the line RH_y; the perspectivity with U as

center transforms Q back to a point H' of the line H_xH_y. We have to show that $H'=H$.

Let H_0 be the trace on the line H_xH_y of PT; let H_1 be the trace of RT; and H' is the trace of UQ.

The complete quadrangle $TRSP$ determines $Q(H_0H_yH_1, H_\infty H_xH)$, and hence (Theorem 3, Chap. VI, Vol. I) in the scale $H_0H_1H_\infty$

$$H_y \cdot H_x = H.$$

The complete quadrangle $TRQU$ determines $Q(H_0H_xH_1, H_\infty H_yH')$, and hence in the scale $H_0H_1H_\infty$

$$H_x \cdot H_y = H'.$$

Since multiplication is commutative, $H=H'$, which proves the theorem.

The reader will find no difficulty in using the construction above to prove that the validity of the theorem of Pappus (§ 36, Vol. I) is necessary and sufficient for the commutative law of multiplication and for Assumption P.

4. Double points of projectivities. DEFINITION. A projective transformation of a real line into itself is said to be *hyperbolic, parabolic*, or *elliptic*,* according as it has two, one, or no double points.

It was proved in § 58, Vol. I, that the determination of the double points of a projective transformation †

$$\text{(1)} \qquad \begin{aligned} \rho x_0' &= a x_0 + b x_1 \\ \rho x_1' &= c x_0 + d x_1 \end{aligned}$$

depends on the solution of the equation

$$\text{(2)} \qquad \rho^2 - (a+d)\rho + \Delta = 0,$$

where $\Delta = ad - bc$. This equation has two real roots if and only if its discriminant

$$(a+d)^2 - 4\Delta$$

is positive. Hence we have

If $\Delta < 0$, *the transformation* (1) *is hyperbolic. For an elliptic or parabolic projectivity* Δ *is always positive.*

* These terms are derived from the corresponding types of conic sections (see § 37). In a complex one-dimensional form a somewhat different terminology is used (cf. § 98).

† In this volume we shall generally write homogeneous coördinates in the form (x_0, x_1), whereas in Vol. I we used (x_1, x_2).

In case the projectivity (1) is an involution, $a = -d$ (§ 54, Vol. I), and hence -4Δ is the discriminant of (2). Hence

An involution is elliptic or hyperbolic according as Δ is positive or negative.

The intimate connection of these theorems with the theory of linear order is evident on comparison with the first sections of Chap. II. A deduction of the corresponding theorems from the intuitive conceptions of order is to be found in Chap. IV of the Geometria Projettiva of Enriques.

EXERCISE

A projectivity for which $\Delta > 0$ is a product of two hyperbolic involutions. A projectivity for which $\Delta < 0$ is a product of three hyperbolic involutions.

5. Complex geometry. Assumption K provides for the solution of many problems of construction which could not be solved in a net of rationality. But even in the real space the fundamental problem of finding the double points of an involution has no general solution.

To see this it is only necessary to set up an involution for which $\Delta > 0$. Take any involution of which two pairs of conjugate points AA' and BB' form a harmonic set $\mathrm{H}(AA', BB')$. If the scale P_0, P_1, P_∞ is chosen so that $A = P_0$, $A' = P_\infty$, $B = P_1$, then $B' = P_{-1}$ and the involution is represented by the bilinear equation (§ 54, Vol. I)

$$xx' = -1.$$

The double points of this involution, if existent, would satisfy the equation

$$x^2 = -1,$$

which has no real roots.

An effect of Assumption K is thus to deny the possibility of solving this problem. If, however, we negate Assumption K and replace it by properly chosen other assumptions, we are led to a geometry in which this problem is always soluble, namely, the geometry of the space in which the geometric number system is isomorphic with the complex number system of analysis. Although this geometry does not have the same relation to the space of external nature as the real geometry, it is extremely important because of its relation to other branches of mathematics.

One way of founding this geometry is to replace Assumption K by another assumption of an equally summary character, namely,

J. *A geometric number system is isomorphic with the complex number system of analysis.*

Since this number system obeys the commutative law of multiplication, the corresponding geometry satisfies Assumption P, and all the theorems of Vol. I apply. Thus, a set of postulates for the complex geometry is A, E, J.

The problem of finding the double points of a one-dimensional projectivity is completely solvable in the complex geometry; for any such projectivity may be represented by the bilinear equation (§ 54, Vol. I)

$$cxx' + dx' - ax - b = 0,$$

and therefore its double points are given by the roots of

$$cx^2 + (d - a)x - b = 0,$$

which exist in the complex number system.

The analogous result holds good for an n-dimensional projectivity. In this case the problem reduces to that of finding the roots of an algebraic equation of the nth degree.

6. Imaginary elements adjoined to a real space. In this connection it is desirable to think of another point of view which we may adopt toward the complex space. Suppose we are working in a real geometry on the basis of A, E, K (or of A, E, H, C, R; see below). It is a theorem about the real number system* that it is contained in a number system (the complex number system) all of whose elements are of the form $ai + b$ where a and b are real and i satisfies the equation

$$i^2 + 1 = 0.$$

Hence it is a theorem about the real space that it is contained in another space which contains the double points of any given involution.

This may be seen in detail as follows: By the theory of homogeneous coördinates the points of a real projective space S are in a correspondence with the ordered tetrads of real numbers (x_0, x_1, x_2, x_3), except $(0, 0, 0, 0)$, such that to each tetrad corresponds one point, and to each point a set of tetrads, given by the expression $(mx_0, mx_1,$

* This same question is discussed from the point of view of a general space and a general field in Chap. IX, Vol. I.

mentioned above, the set of all ordered tetrads of real numbers is contained in the set of all ordered tetrads (z_0, z_1, z_2, z_3) where z_0, z_1, z_2, z_3 are complex numbers.

Let us define a *complex point* as the class of all ordered tetrads of complex numbers of the form

$$(kz_0, kz_1, kz_2, kz_3)$$

where for a given class z_0, z_1, z_2, z_3 are fixed and not all zero and k takes on all complex values different from zero. Let the set of these classes satisfying two independent linear equations

$$\begin{aligned} a_0z_0 + a_1z_1 + a_2z_2 + a_3z_3 &= 0, \\ b_0z_0 + b_1z_1 + b_2z_2 + b_3z_3 &= 0 \end{aligned} \tag{3}$$

be called a *complex line.* With these conventions it is easy to see that the set of all complex points and complex lines satisfies the assumptions A, E, P, and thus the complex points constitute a proper projective space. Let us call this space S_c.

The space S_c contains the set of all complex points of the form

$$(kx_0, kx_1, kx_2, kx_3)$$

where x_0, x_1, x_2, x_3 are all real. Let us call this subset of complex points S_r. If any set of complex points of S_r which satisfy two equations of the form (3) with real coefficients be called a "real line," we have, by reference to the homogeneous coördinate system in S, that the complex points of S_r are in such a one-to-one correspondence with the points of S that to every line in S corresponds a "real line" in S_r, and conversely.

Thus, S_r is a real projective space and is contained in the complex projective space S_c. Obviously S may also be regarded as contained in a complex projective space S′ where S′ consists of the points of S together with the points of S_c which are not in S_r, and where each line of S′ consists of the complex points of S′ which satisfy two equations of the form (3) together with the points of S whose coördinates satisfy the same two equations.

DEFINITION. Points of the real space S are called *real* points, and points of the extended space S′, *complex* points. Points in *S′* but not in *S* are called *imaginary* points.

This discussion of imaginary elements does not require a detailed knowledge or study of the complex number system as such. It is, in fact, a special case of the more general theory in Chap. IX, Vol. I (cf. particularly § 92), which applies to a general projective space. It serves in a large variety of cases where it is sufficient to know merely the *existence* of the complex space S' containing S and satisfying Assumptions A, E, P. It is a logically exact way of stating the point of view of the geometers who used imaginary points before the advent of the modern function theory.

There are problems, however, which require a detailed study of the complex space, and this implies, of course, a study of the complex number system and such geometrical subjects as the theory of chains (see §§ 11, 12, below, and later chapters).

There is a very elegant and historically important method of introducing imaginaries in geometry without the use of coördinates, namely, that due to von Staudt.* It depends essentially on the properties of involutions which are developed in Chap. VIII, Vol. I, and §§ 74–75 of this volume. The reader will find it an excellent exercise to generalize the Von Staudt theory so as to obtain the result stated in Proposition K_2, Chap. IX, Vol. I.

7. Harmonic sequence. We shall now take up a more searching study of the assumptions of the geometry of reals. In Chap. IV, Vol. I, it was proved that every space satisfying Assumptions A, E contains a net of rationality R^3, and that this net is itself a three-space which satisfies not only Assumptions A and E but also Assumption P (Theorem 20). To this rational subspace, therefore, apply all the theorems in Vol. I which do not depend essentially on Assumption H_0. For example, every line of R^3 is a linear net of rationality and may be regarded (with the exception of one point chosen as ∞) as a commutative number system all of whose numbers are expressible as rational combinations of 0 and 1.

Throughout Vol. I we left the character of this net indeterminate. It might contain only a finite number of points or it might contain an infinite number. We propose now to introduce a new assumption which will fix definitely the structure of a net of rationality.

* Cf. K. G. C. von Staudt, Beiträge zur Geometrie der Lage, Nürnberg (1856 and 1857). J. Lüroth, Mathematische Annalen, Vol. VIII (1874), p. 145. Segre, Memorie della R. Accademia delle scienze di Torino (2), Vol. XXXVIII (1886).

and let K_0 be a point of intersection of SH_0 and TH_1. Denote the

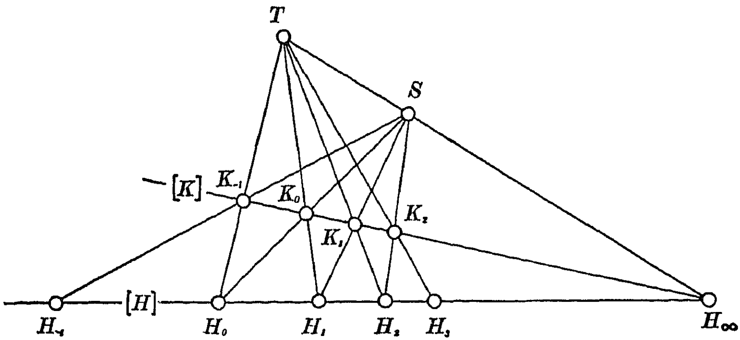


FIG. 3

points of the line h by $[H]$ and those of the line K_0H_∞ by $[K]$, and let Π be a projectivity defined by perspectivities as follows:

$$[H] \overset{S}{\underset{\wedge}{=}} [K] \overset{T}{\underset{\wedge}{=}} [\Pi(H)].$$

The set of points

$$H_0, \quad H_1, \quad H_2, \quad \cdots, \quad H_i, \quad H_{i+1}, \quad \cdots$$

such that $\Pi(H_i) = H_{i+1}$, together with the set

$$\cdots \quad H_{-i-1}, \quad H_{-i}, \quad \cdots, \quad H_{-2}, \quad H_{-1}$$

such that $\Pi(H_{-i-1}) = H_{-i}$, is called a *harmonic sequence.* The point H_∞ is not in the sequence but is called its *limit point.*

The projectivity Π is evidently parabolic and carries H_0 to H_1.

THEOREM 2. *The middle one of any three consecutive* points of a harmonic sequence is the harmonic conjugate of the limit point of the sequence with regard to the other two.*

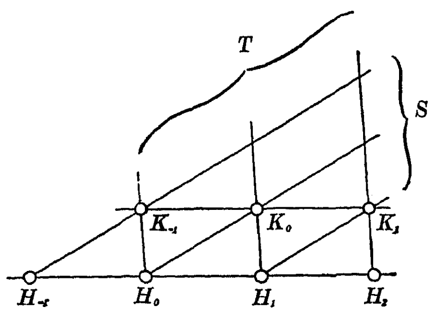


FIG. 4

Proof. By construction we have

$$Q(H_\infty H_i H_{i+1}, \quad H_\infty H_{i+2} H_{i+1}).$$

* This term refers to the subscripts in the notation H_j.

COROLLARY. *All points of a harmonic sequence belong to the same net of rationality.*

THEOREM 3. *Two harmonic sequences determined by H_0, H_1, H_∞ and by M_0, M_1, M_∞ are projective in any projectivity* Π *by which*

$$H_0H_1H_\infty \overline{\wedge} M_0M_1M_\infty.$$

Proof. By Theorem 3, Chap. IV, Vol. I, the projectivity Π transforms harmonic sets of points into harmonic sets.

8. Assumption H. By reference to fig. 3 it is intuitively evident to most observers that in any picture which can be drawn representing points by dots, and lines by marks drawn with the aid of a straightedge, no point H_i which can be accurately marked will ever coincide with $H_j (i \neq j)$. On the other hand, there is nothing in Assumptions A and E to prove that $H_i \neq H_j$, because (Introduction, § 2, Vol. I) these assumptions are all satisfied by the miniature spaces discussed in § 72, Chap. VII, Vol. I, and if the number of points on a line is finite, the sequence must surely repeat itself. Thus we are led to make a further assumption.

ASSUMPTION H.* *If any harmonic sequence exists, not every one contains only a finite number of points.*

The existence of a harmonic sequence determined by any three points follows directly from Assumptions A and E. That any two sequences are projective follows from Theorem 3. Hence Assumption H gives at once

THEOREM 4. *Any three distinct collinear points H_0, H_1, H_∞ determine a harmonic sequence containing an infinite number of points and having H_0 and H_1 as consecutive points and H_∞ as the limit point.*

THEOREM 5. *The principle of duality is valid for all theorems deducible from Assumptions* A, E, H.

Proof. This principle has been proved in Chap. I, Vol. I, for all theorems deducible from A and E. If η_0, η_1, η_∞ are any three planes on a line l, let a line l' meet them in H_0, H_1, H_∞ respectively. The projection by l of the harmonic sequence determined on l' by H_0, H_1, H_∞ is the space dual of a harmonic sequence of points. Since the

evident on the basis of Assumptions A and E alone that the transformation $x' = x + a$ is a parabolic projectivity. Denoting it by α, it is clear that if there is any integer n such that α^n is the identity, then $\alpha^{nk+m} = \alpha^m$, k and m being any integers. Hence, if α has a finite period, there is only a finite number of points in a harmonic sequence, contrary to Assumption H. Hence

THEOREM 6. *A parabolic projectivity never has a finite period. In other words, if of three points determining a harmonic sequence the limit point is taken as ∞ in a scale and two consecutive points as 0 and 1, then the sequence consists of*

$$\begin{array}{ll} 0 & \\ 1 & -1 \\ 1+1=2 & -1-1=-2 \\ 2+1=3 & -2-1=-3 \\ 3+1=4 & -3-1=-4 \\ \vdots & \vdots \end{array}$$

that is, of zero and all positive and negative integers.

COROLLARY 1. *The net of rationality determined by 0, 1, ∞ consists of zero and all numbers of the form $\frac{m}{n}$ where m and n are positive or negative integers.*

Proof. By Theorem 14, Chap. VI, Vol. I, the net of rationality determined by 0, 1, ∞ consists of all numbers obtainable from 0 and 1 by the operations of addition, multiplication, subtraction, and division (excluding division by zero).

COROLLARY 2. *The homogeneous coördinates of any point in a linear planar or spatial net of rationality may be taken as integers.*

Proof. If x_0, x_1, x_2, x_3 are the homogeneous coördinates of a point in the net, they are defined, according to Chap. VII, Vol. I, in terms of the coördinates in certain linear nets. Hence they may be taken in the form 0 or $\frac{m_1}{n_1}$ where m_1 and n_1 are integers. If m is the product of their denominators, mx_0, mx_1, mx_2, mx_3 are integers.

The first of these corollaries enables us to obtain the following simple result with regard to the construction of any point in a net of rationality. Let $H_{\frac{1}{n}}$ be the harmonic conjugate of H_n with regard to H_1 and H_{-1}. The sequence

$$\cdots,\ H_{-\frac{1}{3}},\ H_{-\frac{1}{2}},\ H_{-1},\ H_\infty,\ H_1,\ H_{\frac{1}{2}},\ H_{\frac{1}{3}},\ \cdots$$

is projective (fig. 5) with

$$H_{-3},\ H_{-2},\ H_{-1},\ H_0,\ H_1,\ H_2,\ H_3,\ \cdots$$

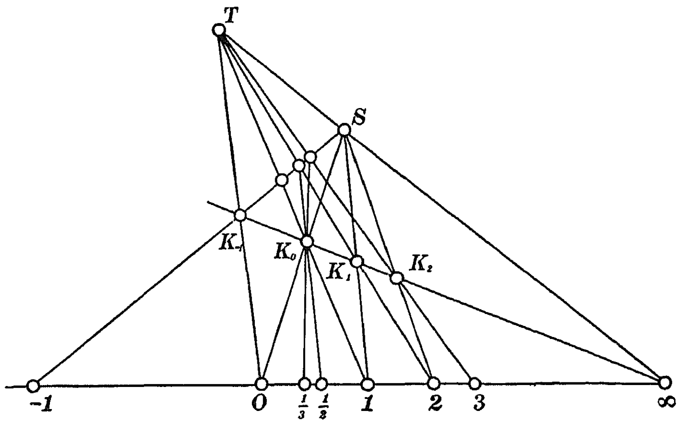


FIG. 5

and therefore must be harmonic. The points H_0, $H_{\frac{1}{n}}$, H_∞ determine a harmonic sequence

$$\cdots,\ H_{-\frac{3}{n}},\ H_{-\frac{2}{n}},\ H_{-\frac{1}{n}},\ H_0,\ H_{\frac{1}{n}},\ H_{\frac{2}{n}},\ H_{\frac{3}{n}},\ \cdots.$$

By Cor. 1, any point of the net of rationality is contained in a sequence of the last variety for some value of n.

9. Order in a net of rationality. DEFINITION. If A and B are points of $\mathsf{R}(H_0H_1H_\infty)$ different from H_∞, A is said to *precede* B with respect to the scale H_0, H_1, H_∞ if and only if the nonhomogeneous coördinate (cf. § 53, Vol. I) of A is less than the nonhomogeneous coördinate of B. If A precedes B, B is said to *follow* A.

From the corresponding properties of the rational numbers there follow at once the fundamental propositions: With respect to the scale H_0, H_1, H_∞, (1) if A precedes B, B does not precede A; (2) if A precedes B and B precedes C, then A precedes C; (3) if A and B are distinct points of $\mathsf{R}(H_0H_1H_\infty)$, then either A precedes B or B precedes A.

The use of the properties of numbers in the argument above and in analogous cases does not imply that our treatment of geometry is dependent on analytical foundations. Every theorem which we employ here is a logical consequence of the assumptions A, E, H alone.

The argument which is involved in the present case may be stated as follows: The coördinates relative to a scale H_0, H_1, H_∞ of the points

$$\cdots, \ H_{-2}, \ H_{-1}, \ H_0, \ H_1, \ H_2, \ \cdots$$

of a harmonic sequence, when combined according to the rules for addition and multiplication given in Chap. VI, Vol. I, satisfy the conditions which are known to characterize the system of positive and negative integers (including zero). From these conditions (the axioms of the system of positive and negative integers) follow theorems which state the order relations among these integers, and also theorems which state the order relations among the rational numbers, the latter being defined in terms of the integers. But by Theorem 6, Cor. 1, the rational numbers are the coördinates of points in $\mathsf{R}(H_0H_1H_\infty)$. Hence the points of $\mathsf{R}(H_0H_1H_\infty)$ satisfy the conditions given above.

It would of course be entirely feasible to make the discussion of order in a net of rationality without the use of coördinates.

DEFINITION. Two subsets, $[A]$ … H_∞) constitute a *cut* (A, B) … nd only if they satisfy the … e net except H_∞ is in $[A]$ … H_∞ every point of $[A]$ precedes every point of $[B]$. If there is a point O in $[A]$ or in $[B]$ such that every point of $[A]$ distinct from O precedes it and every point of $[B]$ distinct from O follows it, the cut is said to be *closed* and to have O as its *cut-point;* otherwise the cut is said to be *open.* The class $[A]$ is said to be the *lower side* and $[B]$ to be the *upper side* of the cut.

With respect to the scale H_0, H_1, H_∞ any point $O\,(O \neq H_\infty)$ of a net $\mathsf{R}(H_0H_1H_\infty)$ determines two sets of points $[A]$ and $[B]$ such that every A precedes or is identical with O and O precedes every B. These sets of points are therefore a closed cut having O as cut-point. Not every cut, however, is closed, for consider the set $[A]$, including all points whose coördinates in a system of nonhomogeneous coördinates having H_∞ as the point ∞ are negative or, if positive, such that their squares are less than 2; and the set $[B]$, including all points whose

* An asterisk at the left of a section number indicates that the section may be omitted on a first reading. We have marked in this manner most of the sections which are not essential to an understanding of the discussion of metric geometry in Chaps. III and IV.

coördinates are positive and have their squares greater than 2. Since no rational number can satisfy the equation

$$x^2 = 2,$$

this equation is not satisfied by the coördinates of any point in the net. The sets $[A]$ and $[B]$ constitute an open cut.

DEFINITION. With respect to the scale H_0, H_1, H_∞, an open cut *precedes* all the points of its upper side and *is preceded by* all points of its lower side. A closed cut *precedes* all the points which its cut-point precedes and *is preceded by* all points by which its cut-point is preceded. A cut (A, B) *precedes* a cut (C, D) if and only if there is a point B preceding a point C.

THEOREM 7. (1) *If a cut (A, B) precedes a cut (C, D), then (C, D) does not precede (A, B).*

(2) *If a cut (A, B) is not the same as the cut (C, D), then either (A, B) precedes (C, D) or (C, D) precedes (A, B), or both cuts are closed and have the same cut-point.*

(3) *If a cut (A, B) precedes a cut (C, D) and (C, D) precedes a cut (E, F), then (A, B) precedes (E, F).*

Proof. These propositions are direct consequences of the definition above and of the corresponding properties of the relation of precedence between points.

DEFINITION. With respect to the scale H_0, H_1, H_∞, a cut (A_1, A_2) is said to be *between* two cuts (B_1, B_2) and (C_1, C_2) in case (B_1, B_2) precedes (A_1, A_2) and (A_1, A_2) precedes (C_1, C_2) or in case (C_1, C_2) precedes (A_1, A_2) and (A_1, A_2) precedes (B_1, B_2). If any one of these cuts is closed, it may be replaced by its corresponding cut-point in this definition. (Thus, for example, any open cut is between any point of its upper side and any point of its lower side.)

An open cut (A, B) is said to be algebraic if there exists an equation,

$$a_0x^n + a_1x^{n-1} + \cdots + a_n = 0,$$

with integral coefficients, and two points A_0, B_0, such that the coördinates of all points of $[A]$ between A_0 and B_0 make the left-hand member of this equation greater than zero and all points of $[B]$ between A_0 and B_0 make it less than zero.* If it is assumed that this equation has a root between A_0 and B_0, this is equivalent to assuming that there exists a point corresponding to the cut (A, B) on the line A_0B_0 but not in the given net.

For the purposes of geometric constructions it would be sufficient to assume the existence of cut-points for all algebraic open cuts (see Chap. IX, Vol. I). For many purposes, indeed, it would be desirable to make the assumption referred to on p. 97, Chap. IV, Vol. I, and which we here put down for reference as Assumption Q.

ASSUMPTION Q. *There is not more than one net of rationality on a line.*

But it is customary in analysis to assume the existence of an irrational number corresponding to every open cut in the system of rationals, and it is convenient in geometry to have a one-to-one correspondence between the points of a line and the system of real numbers. Hence we make the assumption which follows in the next section.

It must not be supposed that in the assumption which follows we are introducing new points in any respect different from those already considered. What we are doing is to postulate that a space is a class of points having certain additional properties. The assumption limits the type of space which we consider; it does not extend the class of points. In this respect our procedure is not parallel to the genetic method of developing the theory of irrational numbers.

EXERCISE

The points of $\mathsf{R}(H_0H_1H_\infty)$, together with the open cuts with respect to the scale H_0, H_1, H_∞, constitute a set $[X]$ of things having the following property: If $[S]$ and $[T]$ are any two subclasses of $[X]$ including all X's and such that every S precedes every T, then there is either an S or a T which precedes all other T's and is preceded by all other S's.

***11. Assumption of continuity.** We shall denote the cut-point of a closed cut (M, N) by $P_{(M,N)}$. In the following assumption it is not stated whether the cuts (A_1, A_2), (B_1, B_2), and (D_1, D_2) are open or closed. If one of them is closed, therefore, the corresponding one of the symbols $P_{(A_1, A_2)}$, $P_{(B_1, B_2)}$, and $P_{(D_1, D_2)}$ must be understood in the sense just defined.

ASSUMPTION C. *If every net of rationality contains an infinity of points, then on one line l in one net $\mathsf{R}(H_0H_1H_\infty)$ there is associated with every open cut (A, B), with respect to the scale H_0, H_1, H_∞, a point $P_{(A,B)}$ which is on l and such that the following conditions are satisfied:*

(1) *If two open cuts (A, B) and (C, D) are distinct, the points $P_{(A,B)}$ and $P_{(C,D)}$ are distinct;*

(2) *If (A_1, A_2) and (B_1, B_2) are any two cuts and (C_1, C_2) any open cut between two points A and B of $\mathsf{R}(H_0H_1H_\infty)$, and if T is a projectivity such that*

$$\mathrm{T}(H_\infty AB) = H_\infty P_{(A_1, A_2)} P_{(B_1, B_2)},$$

then $\mathrm{T}(P_{(C_1, C_2)})$ is a point associated with some cut (D_1, D_2) between (A_1, A_2) and (B_1, B_2).

lows H_0 is called *positive*, and one associated with a cut which precedes H_0 is called *negative*.

THEOREM 8. *The point $P_{(A, B)}$, associated, by Assumption* C, *with an open cut (A, B) of* $\mathsf{R}(H_0H_1H_\infty)$, *is not a point of* $\mathsf{R}(H_0H_1H_\infty)$.

Proof. The associated point could not be H_∞, because there are projectivities of $\mathsf{R}(H_0H_1H_\infty)$ which leave H_∞ invariant and change the given cut into different cuts, and therefore, by Assumption C, change the associated point. Now suppose a point D, distinct from H_∞ but in $\mathsf{R}(H_0H_1H_\infty)$, to be associated with some open cut. Since the given cut is open, there must be a point A between D and the cut. If B is a point on the opposite side of the cut from D, A and B both precede or both follow D with respect to the scale H_0, H_1, H_∞. The transformation which changes every point of l into its harmonic conjugate with regard to H_∞ and D has, when regarded as a transformation of the points of $\mathsf{R}(H_0H_1H_\infty)$ with respect to the scale H_0, H_1, H_∞, the equation

$$x' = 2\,d - x,$$

where d is the coördinate of D. It therefore transforms rational points which follow D into rational points which precede it, and vice versa. Hence A and B are transformed into two points, A' and B', which precede D if A and B follow D, or which follow D if A and B precede D. By Assumption C (2), the point D which is associated with an open cut between A and B is transformed into a point D' associated with a cut between A' and B'. By Assumption C (1), D' is distinct from D, contrary to the hypothesis that D is a fixed point of the transformation.

THEOREM 9. *The points of* $\mathsf{C}(H_0H_1H_\infty)$, *excluding H_∞, form, with reference to the scale in which $H_0 = 0$, $H_1 = 1$, $H_\infty = \infty$, a number system isomorphic with the real number system of analysis.*

Proof. The definitions of Chap. VI, Vol. I, give a meaning to the operations of addition and multiplication for all points of the line l. In that place we derived all the fundamental laws of operation, except

the commutative law of multiplication, on the basis of Assumptions A and E. We have also seen in the present chapter (Theorem 6, Cor. 1) that the coördinates of points in $R(H_0H_1H_\infty)$ are the ordinary rational numbers. Hence it remains to show that the geometric laws of combination as applied to the irrational points of $C(H_0H_1H_\infty)$ are the same as for the ordinary irrational numbers.

The analytic definition of addition of irrational numbers* may be stated as follows: If a and b are two numbers defined by cuts (x_1, y_1) and (x_2, y_2), then $a+b$ is the number defined by the cut (x_1+x_2, y_1+y_2).

To show that our geometric number system satisfies this condition in $C(H_0H_1H_\infty)$, suppose first that a is a rational point of $C(H_0H_1H_\infty)$ and b an irrational point. The projective transformation

$$x' = x + a \tag{4}$$

changes the set of points $[x_2]$ into the set $[x_2+a]$, which is the same as $[x_2+x_1]$. Similarly, it changes $[y_2]$ into $[y_2+y_1]$. Hence, it changes the cut (x_2, y_2) into (x_1+x_2, y_1+y_2), and hence, by Assumption C (2), changes b into a point determined by a cut which lies between every pair x_1+x_2 and y_1+y_2. Therefore b is changed into the point associated with the cut (x_1+x_2, y_1+y_2). But the transform of b is $a+b$. Hence the geometric sum $a+b$ is the number defined by the cut (x_1+x_2, y_1+y_2).

Next, suppose both a and b irrational. The transformation (4) changes $[x_2]$ into the set of irrational points $[x_2+a]$, b into $b+a$, and $[y_2]$ into $[y_2+a]$. By the paragraph above, the cut which defines any x_2+a precedes the cut which defines any y_2+a. Hence, by Assumption C (2), the cut which defines any point x_2+a precedes the cut which defines $b+a$, and this precedes the cut which defines y_2+a. Any point x_1+x_2 of the lower side of the cut (x_1+x_2, y_1+y_2) precedes the cut defining one of the points x_2+a, by the paragraph above, and hence precedes the cut defining $b+a$. Similarly, any point of the upper side of this cut follows the cut defining $b+a$. Hence (x_1+x_2, y_1+y_2) is the cut defining $b+a$. Thus we have identified geometric addition of points in $C(H_0H_1H_\infty)$ with the addition of ordinary real numbers.

be stated as follows: If a and b are positive numbers defined by the cuts (x_1, y_1) and (x_2, y_2), let $[x_1']$ be the set of positive values of x_1. Then ab is the number defined by the cut $(x_1'x_2, y_1y_2)$. If a is negative and b positive, $ab = -(-a)b$. If a is positive and b negative, $ab = -(a(-b))$. If both a and b are negative, $ab = (-a)(-b)$. If $a = 0$ or $b = 0$, $ab = 0$.

Consider the transformation

$$x' = ax.$$

If a is positive and rational while b is positive and irrational, this transforms $[x_2]$ into $[ax_2]$, which is the same as $[x_1'x_2]$. It also transforms b into ab and $[y_2]$ into $[ay_2]$, which is the same as $[y_1y_2]$. Hence, by Assumption C (2), ab is the number associated with $(x_1'x_2, y_1y_2)$.

If both a and b are irrational and positive, we again have $[x_2]$, b, and $[y_2]$ transformed into $[ax_2]$, ab, and $[ay_2]$, where, as in the analogous case of addition, the cut defining ax_2 precedes the cut defining ab, which in turn precedes the cut defining ay_2. Moreover, any $x_1'x_2$ precedes some ax_2, and any y_1y_2 follows some ay_2. Hence, by the same argument as in the case of addition, $(x_1'x_2, y_1y_2)$ is the cut with which ab is associated.

The transformation

$$x' = (-1)x$$

changes the cut (x_1, x_2) defining the irrational number a into the open cut $(-x_2, -x_1)$, which therefore defines an irrational a'. But since $x_1 - x_2$ may be any negative rational and $x_2 - x_1$ may be any positive rational, the sum of a and a', which has been proved to be determined by the cut $(x_1 - x_2, x_2 - x_1)$, must be zero. Hence we have that $(-1)a$ is the irrational $-a$ such that $-a + a = 0$.

The transformation

$$x' = x(-1)$$

is the same as $x' = (-1)x$ for all rational points. Hence, by Assumption C (2), these transformations are the same for all points of $\mathsf{C}(H_0H_1H_\infty)$. Hence, for points of $\mathsf{C}(H_0H_1H_\infty)$, $(-1)x = x(-1)$.

By the associative law of multiplication (which, it is to be remembered, depends only on Assumptions A and E) we have, if a is negative and b positive,

$$ab = -(-a)b,$$

and if both a and b are negative,

$$ab = (-1)(-a)(-1)(-b) = (-a)(-b).$$

COROLLARY. *With respect to a scale in which $H_\infty = \infty$, $H_0 = 0$, $H_1 = 1$, we have $ab = ba$ whenever a and b are in $\mathsf{C}(H_0H_1H_\infty)$.*

THEOREM 10. *Any projectivity which transforms H_0, H_1, and H_∞ into points of the chain $\mathsf{C}(H_0H_1H_\infty)$ transforms any point of the chain into a point of the chain.*

Proof. We have seen that $x' = ax$ and $x' = x + a$, for rational or
es of a, are projectivities which change H_∞ into itself
oints of $\mathsf{C}(H_0H_1H_\infty)$ into points of the chain. The
$1/x$ is a projectivity which interchanges H_∞ and H_0
Vol. I), and by Theorem 9 it changes every point
nto a point of $\mathsf{C}(H_0H_1H_\infty)$.
hap. VI, Vol. I, it follows that
three points of the chain by
hree types. Moreover, any
ormation of $\mathsf{C}(H_0H_1H_\infty)$
ransforms H_0, H_1, H_∞.
, Π and Π', the prod-
hemselves. Hence, by
invariant every point
it would leave invariant
uld be the identity for
me as Π' for all points
ging H_0, H_1, H_∞ into
ct of projectivities of the
ese transform the chain

ng invariant three points of
f the chain invariant.

leaving the given points, say
ectivity such that $\mathrm{P}(B_0B_1B_\infty)$
$= (H_0H_1H_\infty)$. Then $\mathrm{P}\Pi\mathrm{P}^{-1}$ leaves H_0, H_1, H_∞ invariant and hence

leaves all points of the chain invariant, as shown in the proof of the theorem. Hence Π leaves all points of the chain invariant.

COROLLARY 2. *Any projectivity of the chain* $\mathsf{C}(H_0H_1H_\infty)$ *into itself is of the form*

$$\begin{aligned} \rho x_0' &= a x_0 + b x_1, \\ \rho x_1' &= c x_0 + d x_1, \end{aligned} \qquad \begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0,$$

where the coefficients are real numbers.

***12. Chains in general.** DEFINITION. If (A, B) is an open cut in any net of rationality $\mathsf{R}(K_0K_1K_\infty)$ with respect to the scale K_0, K_1, K_∞, let Π be a projectivity transforming $\mathsf{R}(K_0K_1K_\infty)$ into $\mathsf{R}(H_0H_1H_\infty)$ and K_∞ into H_∞. This projectivity transforms (A, B) into a cut (C, D) in $\mathsf{R}(H_0H_1H_\infty)$ with respect to the scale H_0, H_1, H_∞. If X is the point associated by Assumption C with (C, D), the point $\Pi^{-1}(X) = X'$ is called *the irrational cut-point associated with* (A, B).

The point X' is independent of the particular projectivity Π. For let Π' be any projectivity changing (A, B) into a cut (E, F) in $\mathsf{R}(H_0H_1H_\infty)$ with respect to the scale H_0, H_1, H_∞, and let Y be the point associated with (E, F) and $Y' = \Pi'^{-1}(Y)$. Then $\Pi \cdot \Pi'^{-1}$ changes (E, F) into (C, D) and hence, by Assumption C (2), must change Y into X. This can take place only if $Y' = X'$, that is, only if the cut-point X' associated with (A, B) is unique.

By projecting any net of rationality into $\mathsf{R}(H_0H_1H_\infty)$ it is shown that the cut-points associated with it satisfy the conditions stated for the points associated with the cuts of $\mathsf{R}(H_0H_1H_\infty)$ in Assumption C. Hence the theorems of the last section also apply to any chain whatever, a chain being defined as follows:

DEFINITION. The totality of points of a net of rationality $\mathsf{R}(ABC)$, together with all the irrational cut-points defined by open cuts with respect to the scale A, B, C in $\mathsf{R}(ABC)$, is called the *chain* defined by A, B, C and is denoted by $\mathsf{C}(ABC)$. The irrational cut-points are said to be *irrational with respect to* $\mathsf{R}(ABC)$.

Thus we have

THEOREM 11. (1) *The projective transform of a chain is a chain.*

(2) *Every open cut in any net of rationality defines a unique irrational cut-point collinear with, but not in, the net.*

(3) *If two such cuts with respect to the same scale and in the same net are distinct, their cut-points are distinct.*

(4) *If two open cuts are homologous in a projectivity, their cut-points are homologous in the same projectivity.*

(5) *Any projectivity which transforms three points A, B, C into three points of the chain $\mathsf{C}(ABC)$ transforms any point of the chain into a point of the chain.*

THEOREM 12. *There is one and only one chain containing three distinct points of a line.*

Proof. Let A, B, C be the given points. They belong to the chain $\mathsf{C}(ABC)$ into which $\mathsf{C}(H_0H_1H_\infty)$ is transformed by a projectivity such that $H_0H_1H_\infty \,\overline{\wedge}\, ABC$. By Theorem 11 (5) any projectivity such that $ABC \,\overline{\wedge}\, BAC$ transforms all points of $\mathsf{C}(ABC)$ into points of $\mathsf{C}(ABC)$. But by definition such a projectivity transforms $\mathsf{C}(ABC)$ into $\mathsf{C}(BAC)$; hence $\mathsf{C}(BAC)$ is contained in $\mathsf{C}(ABC)$. In like man-
's contained in $\mathsf{C}(BAC)$. Hence $\mathsf{C}(ABC) = \mathsf{C}(BAC) =$

A, B, C to be points of some other chain $\mathsf{C}(PQR)$. By
iectivity such that* $PQRA \,\overline{\wedge}\, QPAR$ changes
nto points of $\mathsf{C}(PQR)$. But by definition it
$\mathsf{C}(QPA)$. Hence $\mathsf{C}(QPA)$ is contained in
ectivity changes $\mathsf{C}(QPA)$ into $\mathsf{C}(PQR)$.
In like manner $\mathsf{C}(QPA) = \mathsf{C}(PBA) =$
$\mathsf{C}(CBA) = \mathsf{C}(ABC)$.

COROLLARY. *A chain contains the irrational cut-point of every open cut in any net of rationality in the chain.*

THEOREM 13. THE FUNDAMENTAL THEOREM OF PROJECTIVITY FOR A CHAIN. *If A, B, C, D are distinct points of a chain and A', B', C' any three distinct points of a line, then for any projectivities giving $(A, B, C, D) \,\overline{\wedge}\, (A', B', C', D')$ and $(A, B, C, D) \,\overline{\wedge}\, (A', B', C', D_1')$ we have $D' = D_1'$.*

Proof. Let Π, Π_1 be the two projectivities mentioned in the theorem. $\Pi_1^{-1}\Pi$ then leaves every point of $\mathsf{C}(ABC)$ fixed; for it leaves every point of $\mathsf{R}(ABC)$ fixed, and hence, by Theorem 11 (4), must leave every irrational cut-point of an open cut in $\mathsf{R}(ABC)$ fixed. But $\Pi_1^{-1}\Pi$ is then the identical transformation as far as the points of

This theorem may also be stated as follows:

Any projective correspondence between the points of two chains is uniquely determined by three pairs of homologous points.

Our list of assumptions for the geometry of reals may now be completed by the following assumption of closure.

ASSUMPTION R. *On at least one line, if there is one there is not more than one chain.*

It follows at once, by Theorem 12, that every line is a chain. It also follows, by an argument strictly analogous to the proof of Theorem 5, that the dual propositions of Assumptions C and R are true. Hence we have

THEOREM 14. *The principle of duality is valid for all theorems deducible from Assumptions* A, E, H, C, R.

***13. Consistency, categoricalness, and independence of the assumptions.** Let us now apply the logical canons explained in the Introduction (Vol. I) to the foregoing set of assumptions.

THEOREM 15. *Assumptions* A, E, H, C, R *are consistent if the real number system of analysis is existent.*

Proof. Consider the class of all ordered tetrads of real numbers (x_0, x_1, x_2, x_3), with the exception of $(0, 0, 0, 0)$. Any class of these ordered tetrads such that if one of its members is (a_0, a_1, a_2, a_3) all its other members are given by the formula (ma_0, ma_1, ma_2, ma_3), where m is any real number not zero, shall be called a point. Any class consisting of all points whose component tetrads satisfy two independent linear homogeneous equations

$$u_0x_0 + u_1x_1 + u_2x_2 + u_3x_3 = 0,$$
$$v_0x_0 + v_1x_1 + v_2x_2 + v_3x_3 = 0$$

shall be called a line. The class of all points and lines so defined satisfy the assumptions A, E, H, C, R (cf. § 4, Vol. I).

THEOREM 16. *Assumptions* A, E, H, C, R *form a categorical set.*

Proof. In Chap. VII, Vol. I, it has been proved that the points of a space satisfying Assumptions A, E, P can be denoted by homogeneous coördinates which are numbers of the geometric number system of Chap. VI, Vol. I. Since P is a logical consequence of A, E, H, C, R (cf. Theorem 12), this result applies here, and by Theorem 9 the

number system in question is isomorphic with the real number system of analysis.

Now if two spaces S_1 and S_2 satisfy A, E, H, C, R, consider a homogeneous coördinate system in each space and let each point of S_1 correspond to that point of S_2 which has the same coördinates. This correspondence is evidently such that if three points of S_1 are collinear, their correspondents in S_2 are collinear.

It is worthy of remark that the above correspondence may be set up in as many ways as there are collineations of S_1 into itself.

THEOREM 17. *Assumptions* A 1, A 2, A 3, E 0, E 1, E 2, E 3, E 3′, H, C, R *are an independent set.*

Proof. The method of proving that a given assumption is not a logical consequence of the other assumptions was explained in the

l. I. Suppose there is given a class of objects

classes of $[x]$. If we call each x a point and

e class of subclasses a line, then each of our assumptions, when thus interpreted, will be either true or false* with respect to this interpretation. If all the assumptions but one are true and the one is false, it cannot be a logical consequence of the others; for a logical consequence of true statements must be true. In the sequel we shall call the objects, x, pseudo-points, and the subclasses of $[x]$ which play the rôle of lines, pseudo-lines.

A 1. The pseudo-points shall be the points of a real projective plane π together with one other point O. The pseudo-lines shall be the lines of π. A 1 is false because there is no pseudo-line containing O. A 2 is true because it is satisfied by the ordinary projective plane. A 3 is true because the only sets of points A, B, C, D, E which satisfy its hypothesis are in π. The only pseudo-plane is π, and there is no pseudo-space. Hence it is evident that E 0, E 1, E 2, E 3 are true and E 3′ is vacuously true. Assumptions H, C, R are evidently true.

* If the hypothesis of a statement is not verified, we regard the statement as true. Following the terminology of E. H. Moore (Transactions of the American Mathematical Society, Vol. III, p. 489), we shall describe statements which are true in this sense as "vacuously true" or "vacuous."

It is possible to put any or all of the assumptions into a form such that they are vacuous for the ordinary real space. For example, Professor Moore has pointed out that A 1 could be replaced by the following proposition, which is vacuous for ordinary space.

$\overline{\text{A}}$ 1. Let A be a point and B be a point. If there is no line which is on A and on B, then $A = B$.

A 2. The pseudo-points shall be the points of a real projective three-space S_3 together with one other pseudo-point O. The pseudo-lines shall be the lines of S_3, each pseudo-line, however, containing O. Thus any two pseudo-points are collinear with O; a pseudo-plane is an ordinary plane together with O; a pseudo-space is S_3 together with O. Hence it is evident that A 2 is false and A 1, A 3, E 0, E 1, E 2, E 3, E 3′ are true. There exist harmonic sequences of pseudo-points, some of which are ordinary harmonic sequences. Hence Assumption H is true. By reference to the definition of a quadrangular set and harmonic conjugate it is clear (because every line contains O) that any pseudo-point P is harmonically conjugate to O with regard to any two pseudo-points which are collinear with P. Hence a linear net of rationality contains all the pseudo-points of a pseudo-line. The operations of addition and multiplication are not unique, however, and hence the definition of order does not apply; there are no open cuts, and Assumptions C and R are vacuously true.

A 3. The pseudo-points shall be the points of a real projective space S_3, with the exception of a single point O. The pseudo-lines shall be the lines of S_3, except that in case of those lines which pass through O the pseudo-lines do not contain O. Clearly A 3 is false whenever the pseudo-points A, B, C, D, E are chosen so that the lines AB and DE meet in O. A 1, A 2, E 0, E 1, E 2, E 3, E 3′ are obviously true. A harmonic sequence and a net of rationality of pseudo-points can be found identical with an ordinary harmonic sequence and net of rationality on any line not passing through O. Hence H, C, and R are also true.

E 0. The pseudo-points shall be the vertices of a tetrahedron, and the pseudo-lines the six pairs of pseudo-points. Thus the pseudo-planes are the trios of pseudo-points, and a pseudo-space consists of all four pseudo-points. A 1 and A 2 are obviously true. A 3 is true because we may have $E=A$ and $D=B$. E 1, E 2, E 3, E 3′ are true. H, C, R are vacuously true.

E 1. There shall be one pseudo-point and no pseudo-line. E 1 is false and all the other assumptions are vacuously true.

E 2. There shall be three pseudo-points and one pseudo-line con-

E 3. The pseudo-points and pseudo-lines shall be the points and lines of a real projective plane. A 1, A 2, A 3, E 0, E 1, E 2, H, C, R are true and E 3′ is vacuous.

E 3′. The pseudo-points and pseudo-lines shall be the points and lines of a real four-dimensional projective space. E 3′ is false and all the other assumptions are true.

H. The pseudo-points and pseudo-lines shall be the points and lines of any modular projective three-space (cf. § 72, Vol. I, and § 16, below). All the assumptions A and E are true, H is false, and C and R are vacuously true.

C. The pseudo-points and pseudo-lines shall be the points and linear nets of rationality of a three-dimensional net of rationality in an ordinary real projective space. All the assumptions are true except C, which is false. R is vacuously true.

R. The pseudo-points and pseudo-lines shall be defined as the points and lines in Theorem 15, the coördinates, however, being elements of the system of ordinary complex numbers. All the assumptions are true except R, which is false.

Assumption C, which is more complicated in its statement than the others, is, however, such that neither of the two statements into which it is separated may be omitted. This result is established in the following theorem:

THEOREM 18. *Assumption* C (1) *is not a consequence of Assumption* C (2) *and all the other assumptions. Assumption* C (2) *is not a consequence of* C (1) *and of the other assumptions even if we add to* C (1) *the following: If a projectivity transforms* H_∞ *into itself and* H_0 *and* H_1 *into points of* $\mathsf{R}(H_0 H_1 H_\infty)$, *and transforms an open cut* (A, B) *into an open cut* (C, D), *it transforms the point associated with* (A, B) *into the point associated with* (C, D).

*Proof.** (1) Any real number x determines a class K_x of numbers of the form $ax + b$ where a and b are any rationals. K_x is the same as K_{ax+b} for all rational values of a and b. Hence, if x and y are two irrationals, K_x and K_y are either identical or mutually exclusive. Thus the class of all real numbers falls into a set of mutually exclusive

* This argument makes use of portions of the theory of classes which could not be treated adequately without a long digression. Hence we assume knowledge of the methods and terminology of this branch of mathematics without further explanation.

numbers of the form $ai+b$, where a and b are rational and $i=\sqrt{-1}$. If we take as pseudo-points and pseudo-lines the points and lines of a three-space based (as in the proof of Theorem 15) on this number system, it is clear that all the assumptions except C are satisfied. If we also take as the pseudo-points H_0, H_1, H_∞ those having the coördinates (0, 1, 0, 0), (1, 1, 0, 0), (1, 0, 0, 0), the net of rationality $\mathsf{R}(H_0 H_1 H_\infty)$ consists of H_∞ and the points whose coördinates are $(x, 1, 0, 0)$, where x is rational. Suppose now that we associate the pseudo-point $(ai+b, 1, 0, 0)$ with every cut in this net which in the ordinary geometry would determine an irrational point $(ak+b, 1, 0, 0)$. Every point is thus associated with an infinity of cuts, contrary to Assumption C (1). Moreover, the cuts with which any point is associated occur between every two pseudo-points and hence between every two cuts of $\mathsf{R}(H_0 H_1 H_\infty)$. Therefore Assumption C (2) remains true in this space.

(2) For the second half of the theorem the pseudo-points and pseudo-lines shall be the points and lines of a three-space based on a commutative number system whose elements are the ordinary rational numbers and all open cuts in the rational numbers. The laws of combination shall be such that addition is precisely the same as for the ordinary number system and multiplication is the same between rationals and rationals or rationals and irrationals, but different between irrationals and irrationals. Thus the product of the numbers associated with two open cuts will not, in general, be the number associated with the cut given by the usual rule. Hence the projective transformation $x' = ax$ will not preserve order relations, and Assumption C (2) must be false. On the other hand, C (1) and the other assumptions are obviously true.

* We do not show how to set up the correspondence. The assumption that this correspondence exists is a weaker form of the assumption used by Zermelo (Mathematische Annalen, Vol. LIX, p. 514) in his proof that any class can be well ordered. Our proof of the second part of the theorem is dependent on the validity of Zermelo's result that the continuum can be well ordered. The whole theorem is therefore subject to the doubts that attach to the Zermelo process because of the lack of explicit methods of setting up the correspondences in question.

The existence of the required new number system can be inferred from Hamel's theorem * that there exists a well-ordered set of real numbers

$$(5) \qquad a_1,\ a_2,\ a_3,\ \cdots,\ a_\omega,\ \cdots$$

such that every real number can be given uniquely by an expression of the form

$$(6) \qquad \alpha_0 + \alpha_1 a_{i_1} + \alpha_2 a_{i_2} + \cdots + \alpha_n a_{i_n},$$

containing only a finite number of terms, the α's all being rational. The ordinary rules of combination for cuts determine a multiplication table for the a's; that is, a set of rules of the form

$$(7) \qquad a_i a_j = \beta_0 + \beta_1 a_{k_1} + \beta_2 a_{k_2} + \cdots + \beta_m a_{k_m},$$

where the β's are rational. The laws of combination for the number system in general may now be stated as follows: Express the two numbers to be added or multiplied in the form (6); add or multiply by the rules for addition and multiplication of polynomials, reducing the result in the case of multiplication by means of the multiplication table for the a's.

Now suppose we denote by

$$(8) \qquad a'_1,\ a'_2,\ \cdots,\ a'_\omega,\ \cdots$$

the same set of numbers $[a]$ arranged in a different order of the same type as (5). Such an order would be obtained, for example, by interchanging a_1 and a_2 and leaving the other a's unaltered. There is therefore a one-to-one correspondence in which every a_i corresponds to the a'_i having the same subscript. Moreover, since the set of all a's includes the same elements as the set of all a''s, every real number is expressible in the form

$$(9) \qquad \alpha_0 + \alpha_1 a'_{i_1} + \alpha_2 a'_{i_2} + \cdots + \alpha_n a'_{i_n}.$$

A new law of multiplication, which we shall denote by $\times$, is now defined by setting up a multiplication table for the a''s according to the rule that

$$(10) \qquad a'_i \times a'_j = \alpha_0 + \alpha_1 a'_{i_1} + \cdots + \alpha_n a'_{i_n}$$

whenever

$$(11) \qquad a_i a_j = \alpha_0 + \alpha_1 a_{i_1} + \cdots + \alpha_n a_{i_n}.$$

* Mathematische Annalen, Vol. LX, p. 459.

numbers is obtained by expressing each in the form (9), multiplying according to the rule for polynomials, and reducing by the multiplication table for the a''s.

Since the set of all expressions of the form

$$\alpha_0 + \alpha_1 a_{i_1} + \alpha_2 a_{i_2} + \cdots$$

forms a number system, the set of all expressions of the form

$$\alpha_0 + \alpha_1 a'_{i_1} + \alpha_2 a'_{i_2} + \cdots$$

forms a number system isomorphic with the first. For if we let each a_i correspond to the a'_i with the same subscript, the sum of any two elements of the first number system corresponds, by definition, to the sum of the corresponding two elements in the second number system. Similarly for the product of a rational by a rational or of a rational by an irrational. The product of two irrationals in the first system corresponds to the product of two irrationals in the second, because the two polynomials in the a's are multiplied by the same rules as the two in the a''s, and are also reduced by corresponding entries in the respective multiplication tables.

We may insure that the two number systems shall be distinct by selecting the a's, in the first place, so that $a_1 = \sqrt{2}$ and $a_2 = \sqrt{3}$, and then choosing the a''s so that $a'_1 = a_2$.

***14. Foundations of the complex geometry.** Let us add to Assumptions A, E, H, C the following assumption:

ASSUMPTION $\overline{\text{R}}$. *On some line, l, not all points belong to the same chain.*

Let P_0, P_1, P_∞ be three points of l. The geometric number system determined by the method of Chap. VI, Vol. I, by the scale P_0, P_1, P_∞ is commutative for all the points in the chain $\mathsf{C}(P_0P_1P_\infty)$ but not necessarily for other points. However, it is clear, without assuming the commutativity of multiplication, that

$$x' = x^{-1},\ \ x' = x + a,\ \ x' = ax,\ \ x' = xa \quad (a = \text{constant})$$

define projectivities. For $x' = x^{-1}$ this follows from § 54, Vol. I; for $x' = x + a$ it reduces to Theorem 2, Chap. VI, Vol. I; and for the other two cases, to Theorem 4, Chap. VI, Vol. I.

Let J be any point of l not in $\mathsf{C}(P_0P_1P_\infty)$, and let $[X]$ be the set of all points in $\mathsf{C}(P_0P_1P_\infty)$. Then, by Theorem 11 (1), the set of points

$[X+J]$ is a chain. This chain has no point except P_∞ in commo with $\mathsf{C}(P_0P_1P_\infty)$, because, if $X+J=X'\neq P_\infty$, it would follow tha $X'-X=J$, and thus J would be a point of $\mathsf{C}(P_0P_1P_\infty)$. Let us denot the chain $[X+J]$ by C'.

In order to continue this argument we need the following assump tion of closure:

ASSUMPTION I. *Through a point P of any chain C of the line and any point J on l but not in C, there is not more than one chai of l which has no other point than P in common with C.*

Now let P be any point of l not in $\mathsf{C}(P_0P_1P_\infty)$ or C'. Such point exist, because, for example, the chain $\mathsf{C}(P_0P_1J)$ does not coincide wit $\mathsf{C}(P_0P_1P_\infty)$ or C'. The chain $\mathsf{C}(PJP_\infty)$ has, by Assumption I, a poin different from P_∞ in common with $\mathsf{C}(P_0P_1P_\infty)$. Let X_1 be this poin In case $X_1\neq P_0$, the projectivity

$$X'=X+J(P_1-X_1^{-1}\cdot X) \tag{12}$$

transforms P_0 into J, X_1 into itself, and P_∞ into itself. Hence it trans forms $\mathsf{C}(P_0P_1P_\infty)=\mathsf{C}(P_0X_1P_\infty)$ into $\mathsf{C}(JX_1P_\infty)$. Hence every point $\mathsf{C}(JX_1P_\infty)$, and in particular P, is of the form $X+JX''$, where X an X'' are in $[X]$. If $X_1=P_0$, the projectivity

$$X'=JX \tag{13}$$

transforms $\mathsf{C}(P_0,\ P_1,\ P_\infty)$ into $\mathsf{C}(P_0JP_\infty)$, which contains P. Hence, i this case P is of the form JX. Thus we have

LEMMA 1. *Every point of the line l is expressible in the form $A+J$ where A and B are in $\mathsf{C}(P_0P_1P_\infty)$.*

LEMMA 2. *Two points $A+JB$ and $A'+JB'$, where A, B, A', B' a in $\mathsf{C}(P_0P_1P_\infty)$, are identical if and only if $A=A'$ and $B=B'$.*

For if $B\neq B'$, $A+JB=A'+JB'$ implies $J=(A'-A)(B-B')^-$ and thus J would be in $\mathsf{C}(P_0P_1P_\infty)$; and if $B=B'$, it implies directl that $A=A'$.

Each of the projectivities $X' = (P_1 - J)X$ and $X' = X(P_1 - J)$ transforms $\mathsf{C}(P_0P_1P_\infty)$ into $\mathsf{C}(P_0(P_1 - J)P_\infty)$. Hence, if A be any point of $\mathsf{C}(P_0P_1P_\infty)$,

$$A(P_1 - J) = (P_1 - J)A'',$$

where A'' is also in $\mathsf{C}(P_0P_1P_\infty)$. By the distributive law (Theorem 5, Chap. VI, Vol. I) it follows that

$$A - AJ = A'' - JA''.$$

By (14), this reduces to

$$A - JA' = A'' - JA''.$$

By Lemma 2, it follows that $A = A'' = A'$. Hence $AJ = JA$. From this we can deduce, by the elementary laws of operation,

$$\begin{aligned}(A + JB)(C + JD) &= A(C + JD) + JB(C + JD)\\ &= AC + AJD + JBC + JBJD\\ &= CA + CJB + JDA + JDJB\\ &= C(A + JB) + JD(A + JB)\\ &= (C + JD)(A + JB).\end{aligned}$$

Hence the geometric number system determined by any scale on l is commutative. Since chains are transformed into chains by any projective transformation, it follows that the geometric number system determined by any scale on any line in a space satisfying A, E, H, C, $\overline{\mathrm{R}}$, I satisfies the commutative law of multiplication. Hence, by Theorem 1,

THEOREM 19. *Assumption* P *is satisfied in any space satisfying Assumptions* A, E, H, C, $\overline{\mathrm{R}}$, I.

Since every point in the geometric number system is expressible in the form $A + JB$, we have

$$J^2 = A_0 + JB_0, \tag{15}$$

where A_0 and B_0 are in $\mathsf{C}(P_0P_1P_\infty)$. Thus J is one of the double points of the involution

$$XX' - \tfrac{1}{2}B_0(X + X') - A_0 = 0, \tag{16}$$

which transforms $\mathsf{C}(P_0P_1P_\infty)$ into itself. Any two points of $\mathsf{C}(P_0P_1P_\infty)$ which are conjugate in this involution may be transformed projectively into P_0 and P_∞ by a transformation which carries $\mathsf{C}(P_0P_1P_\infty)$ into itself. This reduces the involution to

where A must be negative relatively to the scale $P_0P_1P_\infty$, since the double points are not in $C(P_0P_1P_\infty)$. The transformation $X = \sqrt{-A}X'$ now reduces (17) to

$$XX' = -P_1$$

and thus transforms J to a point satisfying the equation

$$J^2 = -P.$$

Hence we have

THEOREM 20. *The geometric number system in any space satisfying Assumptions* A, E, H, C, $\overline{\text{R}}$, I *is isomorphic with the complex number system of analysis, i.e. with the system of numbers $a + ib$, where $i^2 = -1$ and a and b are real.*

***15. Ordered projective spaces.** There is an important class of projective spaces which may be referred to as the *ordered projective spaces* and which are characterized by the Assumptions S given below. This class of spaces includes the rational and real projective spaces and many others. The set of assumptions, A, E, S, is not categorical, but it may be made so by adding a suitable continuity assumption or by some other assumption of closure.

These assumptions introduce a new class of undefined elements, called *senses*,* in addition to the points and lines which are the undefined elements of Assumptions A and E. The senses are denoted by symbols of the form $S(ABC)$, where A, B, C denote points.†

S 1. *For any three distinct collinear points A, B, C there is a sense $S(ABC)$.*

S 2. *For any three distinct collinear points there is not more than one sense $S(ABC)$.*

S 3. $S(ABC) = S(BCA)$.

S 4. $S(ABC) \neq S(ACB)$.

S 5. *If $S(ABC) = S(A'B'C')$ and $S(A'B'C') = S(A''B''C'')$, then $S(ABC) = S(A''B''C'')$.*

S 6. *If $S(ABO) = S(BCO)$, then $S(ABO) = S(ACO)$.*

S 7. *If OA and OB are distinct lines, and $S(OAA_1) = S(OAA_2)$ and $OAA_1A_2 \overline{\wedge} OBB_1B_2$, then $S(OBB_1) = S(OBB_2)$.*

* Sets of assumptions more or less related to these have been given by A. R. Schweitzer, American Journal of Mathematics, Vol. XXXI, p. 365, and A. N. Whitehead, The Axioms of Projective Geometry, Cambridge Tracts, Cambridge, 1906.

† With respect to the intuitional basis of these assumptions, cf. figs. 6–12, Chap. II.

If $S(ABC)$ be identified with the sense-class which is discussed below in § 19, Chap. II, it will be seen that S 1 and S 2 are immediately verified and S 3, $\cdots$, S 7 reduce to Theorems 2–6, Chap. II. This shows that the assumptions S are satisfied by a rational or a real projective space.

These assumptions are capable, as is shown in Chap. II, of serving as a basis for a very complete discussion of geometric order relations. Assumption P is not a consequence of A, E, S alone.

EXERCISES

1. Prove that Assumption H is a consequence of A, E, and S.

2. Prove that with a proper definition of the symbol $<$ (less than) the geometric number system in an ordered projective space satisfies the following conditions:

(1) If a and b are distinct numbers, $a < b$ or $b < a$.
(2) If $a < b$, then $a \neq b$.
(3) If $a < b$ and $b < c$, then $a < c$.
(4) If $a < b$, there exists a number, x, such that $a < x$ and $x < b$.
(5) If $0 < a$, then $b < a + b$ for every b.
(6) If $0 < a$ and $0 < b$, then $0 < a \cdot b$.

(Cf. E. V. Huntington, Transactions of the American Mathematical Society, Vol. VI (1905), p. 17.)

3. Introduce an assumption of continuity and with this assumption and A, E, S prove Assumption P.

4. Prove that P is not a consequence of A, E, S alone.

***16. Modular projective spaces.** We have seen (§ 7) that, in any space satisfying Assumptions A and E, any two harmonic sequences are projective. Hence, if one harmonic sequence contains an infinity of points, every such sequence contains an infinity of points, and by § 8 these points are in one-to-one reciprocal correspondence with the ordinary rational numbers. On the other hand, if one harmonic sequence contains a finite number of points, every other harmonic sequence in the same space contains the same finite number of points. Hence the spaces satisfying Assumptions A and E fall into two classes — those satisfying Assumption H and those satisfying the following:

ASSUMPTION $\overline{\text{H}}$. *If any harmonic sequence exists, at least one contains only a finite number of points.*

ing H *nonmodular*.

It follows, just as in Theorem 5, that the principle of duality is true for any modular space.

Let Π be any parabolic projectivity on a line, and let H_∞ be its invariant point. If H_0 be any other point of the line, the points

$$\cdots \Pi^{-2}(H_0),\ \Pi^{-1}(H_0),\ H_0,\ \Pi(H_0),\ \Pi^2(H_0) \cdots$$

form a harmonic sequence, by definition. If this is to contain only a finite number of points, there must be some positive integer n such that $\Pi^n(H_0) = \Pi^m(H_0)$, where m is zero or a positive integer less than n. If $n - m = k$, we have

$$\Pi^k(\Pi^m(H_0)) = \Pi^n(H_0),$$

and hence
$$\Pi^k = 1.$$

Hence all the points of the harmonic sequence are contained in the set

$$H_0,\ \Pi(H_0),\ \cdots,\ \Pi^{k-1}(H_0).$$

In case k is not a prime number, that is, if there exist two positive integers, k_1, k_2, different from unity such that $k = k_1 \cdot k_2$, let us consider the parabolic projectivity Π^{k_2}. The points

$$H_0,\ \Pi^{k_2}(H_0),\ \Pi^{2k_2}(H_0),\ \cdots,\ \Pi^{(k_1-1)k_2}(H_0)$$

satisfy the definition of a harmonic sequence. Since any two harmonic sequences contain the same number of points, it follows that the given sequence could not have contained more than k_1 points. In case k_1 breaks up into two factors, the same argument shows that the given harmonic sequence could not contain a number of points larger than either factor. This process can be repeated only a finite number of times and can stop only when we arrive at a prime number. Hence we have

THEOREM 21. *The number of points in a harmonic sequence is prime. The points of a harmonic sequence may be denoted by*

$$H_0,\ \Pi(H_0),\ \cdots,\ \Pi^{p-1}(H_0),$$

where Π is a parabolic projectivity. The period, p, of any parabolic projectivity is a prime number.

With reference to a scale in which $H_0 = 0$, $\Pi(H_0) = 1$, and the limit point of the harmonic sequence is ∞, Π has the equation

$$x' = x + 1.$$

Hence the coördinates of the points in the harmonic sequence are

$$0,\ 1,\ 2,\ \cdots,\ p-1,$$

respectively, where 2 represents $1+1$, 3 represents $2+1$, etc. Since $\Pi^p = 1$, we must have that $p = 0$, $p+1 = 1$, $np+k = k$, etc. In other words, the coördinates of the points in a harmonic sequence are elements of the field obtained by reducing the integers modulo p, as explained in § 72, Vol. I.

By Theorem 14, Chap. VI, Vol. I, the net of rationality determined by the points whose coördinates are 0, 1, ∞ consists of the point ∞ and all points whose coördinates are obtainable from 0 and 1 by the operations of addition, subtraction, multiplication, and division (except division by zero). Since all numbers of this sort are contained in the set

$$0,\ 1,\ \cdots,\ p-1,$$

we have

THEOREM 22. *The number of points in a net of rationality in a modular space is $p+1$, p being a prime number constant for the space in question.*

Obviously, if Assumption Q (§ 10) be added to the set A, E, $\overline{\mathrm{H}}$, the number of points on any line must be $p+1$, p being prime. A space satisfying A, E, $\overline{\mathrm{H}}$ shall be called a *rational modular space.* The problem of finding the double points of a projectivity in a rational modular space of one or more dimensions leads to the consideration of modular spaces bearing a relation to the rational ones analogous to the relation which the complex geometry bears to the real geometry. The existence of such spaces follows from the considerations in Chap. IX, Vol. I (Propositions K_2 and K_n). The geometric number systems for such spaces may be finite* (Galois fields) or infinite.†

* E. H. Moore, The Subgroups of the Generalized Finite Modular Group, Decennial publications of The University of Chicago, Vol. IX (1903), pp. 141–190; L. E. Dickson, Linear Groups, Chap. I.

† L. E. Dickson, Transactions of the American Mathematical Society, Vol. VIII (1907), p. 389. See also the article by E. Steinitz referred to in § 92, Vol. I.

17. Recapitulation. The various groupings of assumptions which we have considered thus far may be resumed as follows: A space satisfying Assumptions

A, E	is a general projective space;
A, E, P	is a proper projective space;
A, E, H	is a nonmodular projective space;
A, E, $\overline{\text{H}}$	is a modular projective space;
A, E, S	is an ordered projective space;
A, E, $\overline{\text{H}}$, Q	is a rational modular projective space;
A, E, H, Q	is a rational nonmodular projective space;
A, E, H, C, R or A, E, K	is a real projective space;
A, E, H, C, $\overline{\text{R}}$, I or A, E, J	is a complex projective space.

The first six sets of assumptions are not, and the remaining ones are, categorical. The set of theorems deducible from any one of these sets of assumptions is called a projective geometry, and the various geometries may be distinguished by the adjectives applied above to the corresponding spaces.

CHAPTER II

ELEMENTARY THEOREMS ON ORDER

18. Direct and opposite projectivities on a line. In § 9 a point A was said to precede a point B relative to a scale P_0, P_1, P_∞ if the coördinate of A in this scale was less than the coördinate of B. Supposing the coördinate of A to be a and that of B to be b, the projectivity changing P_0 to A and P_1 to B and leaving P_∞ fixed has the equation

$$x' = (b - a)x + a. \tag{1}$$

In this transformation the coefficient of x is positive if and only if A precedes B. But the transformations of the form

$$x' = \alpha x + \beta, \tag{2}$$

where α is positive, evidently form a group. This group is a subgroup of the group of all projectivities leaving P_∞ invariant, for the latter group contains all transformations (2) for which $\alpha \neq 0$.

The group of transformations (2) for which α is positive is, by what we have just seen, such that whenever a pair of points A and B are transformed to A' and B' respectively, A precedes B if and only if A' precedes B'. The discussion of order relative to a scale could therefore be based on the theory of this group.

The order relations defined by means of this group have all, however, a special relation to the point P_∞, and they can all be derived by specialization from a more general relation defined by means of a more extensive group. We shall therefore enter first into the discussion of this larger group, and afterwards (§ 23) show how to derive the relations of "precede" and "follow" from the general notion of "sense." The definitions for the general case, like those for the special one, will be seen to depend simply on the distinction between positive and negative numbers.

A projective transformation of a line may be written in the form

$$\begin{aligned} x_0' &= a_{00}x_0 + a_{01}x_1, \\ x_1' &= a_{10}x_0 + a_{11}x_1, \end{aligned} \qquad \Delta = \begin{vmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{vmatrix} \neq 0, \tag{3}$$

where the a_{ij}'s are numbers of the geometric number system

sumption H, the a_{ij}'s may be taken (Theorem 6, Cor. 2, Chap. I) a integers. The discussion which follows is valid on either hypothesis.

Definition. The projectivities of the form (3) for which $\Delta >$ are called *direct*, and those for which $\Delta < 0$ are called *opposite.*

Since the determinant of the product of two transformations (3 is the product of the determinants, the direct projectivities form subgroup of the projective group. The same transformation (3) canno be both direct and opposite, for two transformations (3) are identica only if the coefficients of one are obtainable from those of the othe by multiplying them all by the same constant ρ; but this merel changes Δ into $\rho^2\Delta$.

In form, the definition is dependent on the choice of the coördinat system which is used in equations (3). Actually, however, the defi nition is independent of the coördinate system, for if a given proje tivity has a positive Δ with respect to one scale, it has a positive with respect to every scale. This may be proved as follows:

Let the fundamental points of the scale to which the coördinate in (3) refer be P_0, P_1, P_∞, and let Q_0, Q_1, Q_∞ be the fundamental points any other scale. By § 56, Vol. I, the coördinates y_0, y_1 of any point with respect to any scale Q_0, Q_1, Q_∞ are such that $y_1/y_0 = R(Q_\infty Q_0, Q_1 R$ Suppose that, relative to the scale P_0, P_1, P_∞, the projectivity whic transforms Q_0, Q_1, Q_∞ to P_0, P_1, P_∞ respectively has the equations

$$(4) \qquad \begin{aligned} y_0 &= b_{00}x_0 + b_{01}x_1, \\ y_1 &= b_{10}x_0 + b_{11}x_1, \end{aligned} \qquad \begin{vmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{vmatrix} = D \neq 0.$$

Thus any point R whose coördinates relative to the scale $P_0, P_1,$ are (x_0, x_1) is transformed by this projectivity to a point R' whos coördinates relative to the scale P_0, P_1, P_∞ are (y_0, y_1).

Since cross ratios are unaltered by projective transformations,

$$R(Q_\infty Q_0, Q_1 R) = R(P_\infty P_0, P_1 R') = \frac{y_1}{y_0}.$$

Hence it follows that *if x_0 and x_1 are the coördinates of any point relative to the scale P_0, P_1, P_∞, the corresponding values of y_0 and y_1 give*

* It is, in fact, valid in any space satisfying Assumptions A, E, S, P. The purel ordinal theorems are indeed valid in any ordered projective space (§ 15), but thos regarding involutions, conic sections, etc. necessarily involve Assumption P als Cf. the fine print at the end of § 19.

by (4) *are the coördinates of* R *relative to the scale* Q_0, Q_1, Q_∞. Let us indicate (4) by $(y_0, y_1) = T(x_0, x_1)$, and (3) by $(x_0', x_1') = S(x_0, x_1)$.

Now a direct transformation (3) carries a point whose coördinates relative to the scale P_0, P_1, P_∞ are (x_0, x_1) into one whose coördinates relative to the same scale are (x_0', x_1'), where $(x_0', x_1') = S(x_0, x_1)$. The coördinates of these two points relative to the scale Q_0, Q_1, Q_∞ are $(y_0, y_1) = T(x_0, x_1)$ and $(y_0', y_1') = T(x_0', x_1')$ respectively. Hence, by substitution,

$$(y_0', y_1') = T(S(x_0, x_1)) = T(S(T^{-1}(y_0, y_1))),$$

or

$$(y_0', y_1') = TST^{-1}(y_0, y_1),$$

where T^{-1} indicates, as usual, the inverse of T. The determinant of the transformation TST^{-1} is

$$\Delta' = D\Delta\frac{K^2}{D},$$

where K is real (or rational), and Δ' therefore has the same sign as Δ. Thus the definition of a direct projectivity is independent of the choice of the coördinate system.

This result can be put in another form which is important in the sequel:

DEFINITION. Two figures are said to be *conjugate under* or *equivalent with respect to* a group of transformations if and only if there exists a transformation of the group carrying one of the figures into the other.

THEOREM 1. *If two sets of points are conjugate under the group of direct projectivities on a line, so are also the two sets of points into which they are transformed by any projectivity of the line.*

Proof. Let S be a direct projectivity changing a set of points $[A]$ into a set of points $[B]$, and let T be any other projectivity on the line, and let $T(A) = A'$ and $T(B) = B'$. Since $T^{-1}(A') = A$, $S(A) = B$, and $T(B) = B'$, it follows that $TST^{-1}(A') = B'$. But the discussion above shows that TST^{-1} is a direct projectivity. Hence $[A']$ and $[B']$ are conjugate under the group of direct projectivities, as was to be proved.

According to the definition in § 75, Vol. I (see also § 39, below), the group

EXERCISES

1. Within the field of all real numbers the positive numbers may be define as those numbers different from zero which possess square roots. Generaliz this definition to other fields, and thus generalize the definitions of direc projectivities. In each case determine how far the theorems on sense an order in the following sections can be generalized (cf. § 72, Vol. I).

2. The group of projectivities which transform a net of rationality int itself has a self-conjugate subgroup consisting of those transformations whic are products of pairs of involutions having their double points in the net o rationality. This group contains all projectivities for which the determinan is the square of a rational number.

***3.** Work out a definition and theory of the group of direct projectivitie independent of the use of coördinates. This may be done by the aid o theorems in Chap. VIII, Vol. I (cf. §§ 69 and 70, below).

19. The two sense-classes on a line. DEFINITION. Let A_0, B_0, C_0 b any three distinct points of a line. The class of all ordered* triad of points ABC on the line, such that the projectivities

$$A_0B_0C_0 \barwedge ABC$$

are direct, is called a *sense-class* and is denoted by $S(A_0B_0C_0)$. Tw ordered triads in the same sense-class are said *to have the same sens* or *to be in the same sense.* Two collinear ordered triads not in the sam sense-class are said *to have opposite senses* or *to be in opposite sense*

One sense-class chosen arbitrarily may be referred to by a particular nam as *right-handed*, *clockwise*, *positive*, etc.†

The term "sense," standing by itself, might have been defined as follows "The senses are any set of objects in one-to-one and reciprocal correspondenc with the sense classes." This is analogous to the definition of a vector give in § 42. When there is question only of one line, any two objects whateve may serve as the two senses — for example, the signs + and −. This agree with the definition of sense as "the sign of a certain determinant." Whe dealing with more than one line, it is no longer correct to say that there ar two senses; there are, in fact, two senses for each line.

* "Order," here, is a logical rather than a geometrical term, just as in the defi nition of "throw" (§ 23, Vol. I). It is a device for distinguishing the elements o a set. For example, when we say that ABC cannot be transformed into ACB b any transformation of a given group, it is a way of saying that the group contain no transformation changing A into A, B into C, and C into B.

† A partial list of references on the notion of sense in one and more dimension would include: Möbius, Barycentrische Calcul, note in § 140; Gauss, Werke, Vol VIII, p. 248; von Staudt, Beiträge zur Geometrie der Lage, §§ 3, 14; Study, Archi

When one adopts, as we do, the symbol $S(ABC)$ to stand for a sense-class, there is no occasion for attaching a separate meaning to the word "sense." It may be regarded as an incomplete symbol,* like the $\frac{d}{dx}$ in the $\frac{dy}{dx}$ of the calculus.

THEOREM 2. *If the ordered triad ABC is in the sense-class $S(A_0B_0C_0)$, then $S(ABC) = S(A_0B_0C_0)$. If $S(ABC) = S(A'B'C')$ and $S(A'B'C') = S(A''B''C'')$, then $S(ABC) = S(A''B''C'')$.*

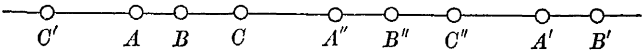


FIG. 6

Proof. Both statements are consequences of the fact that the direct projectivities form a group.

THEOREM 3. *If $S(ABC) \neq S(A'B'C')$ and $S(A'B'C') \neq S(A''B''C'')$, then $S(ABC) = S(A''B''C'')$.*

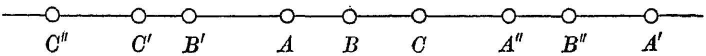


FIG. 7

Proof. If $S(ABC) \neq S(A'B'C')$, the projectivity $ABC \barwedge A'B'C'$ is opposite. Hence the theorem follows from the fact that the product of two opposite projectivities is direct.

COROLLARY. *There are two and only two sense-classes on a line.*

THEOREM 4. *If A, B, C are distinct collinear points, $S(ABC) = S(BCA)$ and $S(ABC) \neq S(ACB)$.*†

Proof. Let A, B, C be taken as (1, 1), (1, 0), (0, 1) respectively. Then

$$x_0' = x_1,$$
$$x_1' = x_0$$

is an opposite projectivity interchanging B and C and leaving A invariant. Hence $S(ABC) \neq S(ACB)$. In like manner, we can prove that $S(ACB) \neq S(BCA)$. It follows, by Theorem 3, that $S(ABC) = S(BCA)$.

* The term "incomplete symbol" appears in Whitehead and Russell's Principia Mathematica, Vol. I, Chap. III, of the Introduction, together with a discussion of its logical significance.

† This may be expressed by the phrase "Sense is preserved by even and altered by odd permutations." A *transposition* is a permutation in which two and only two

THEOREM 5. *If* $S(ABD) = S(BCD)$, *then* $S(ABD) = S(ACD)$.

Proof. Choose the coördinates so that $D = (0, 1)$, $A = (1, 0)$, $B = (1, 1)$. The transformation of ABD to BCD may be written in the form

$$x_0' = x_0,$$
$$x_1' = x_0 + ax_1,$$

FIG. 8

because (0, 1) is invariant and (1, 0) goes to (1, 1). This transformation will be direct if and only if $a > 0$. The point C, being the transform of (1, 1), is $(1, 1 + a)$. The transformation carrying ABD to ACD is

$$x' = x_0,$$
$$x_1' = (1 + a)x_1,$$

which is direct because $(1 + a) > 0$.

As an immediate consequence of Theorem 1 we have

THEOREM 6. *If* $S(ABC) = S(A_1B_1C_1)$ *and* $ABCA_1B_1C_1 \barwedge A'B'C'A_1'B_1'C_1'$, *then*

$$S(A'B'C') = S(A_1'B_1'C_1').$$

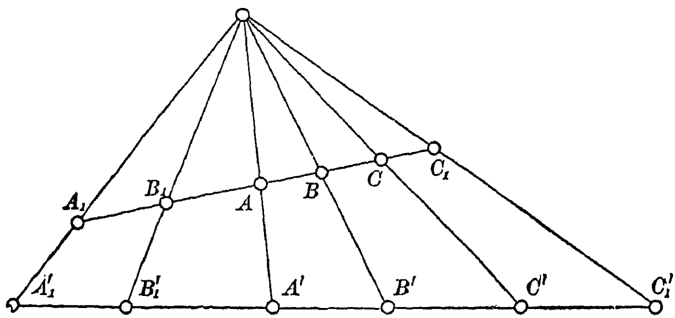


FIG. 9

Theorems 2–6 contain the propositions given in § 15, Chap. I, as Assumptions S. Theorem 6 is slightly more general than S 7 but is directly deducible from it. The developments of the following sections will be based entirely on these propositions, and hence belong to the theory of any ordered projective space, except where reference is made to figures whose existence depends on Assumption P Theorems of the latter sort hold in any space satisfying A, E, P, S.

These propositions have the advantage, as assumptions, of corresponding to some of our simplest intuitions with regard to the linear order relations. The reader may verify this by constructing the figures to which they correspond (cf. figs. 6–9). Each proposition will be found to correspond to a number of

$$1231'2'3' \,\overline{\wedge}\, ABCA'B'C',$$

then the ordered triad 123 is said *to have the same sense as* $1'2'3'$ if and only if $S(ABC) = S(A'B'C')$. The set of all ordered triads having the same sense as 123 is called a *sense-class* and denoted by $S(123)$.

In view of Theorem 6 this definition is independent of the choice of the points A, B, C, A', B', C'. It is an immediate corollary of the definition that the plane and space duals of Theorems 2–6 all hold good (cf. figs. 10 and 13).

By the definition of a point conic there is a one-to-one correspondence between the points $[P]$ of the conic and the lines joining them to a fixed point P_0 of the conic. We now define any statement in

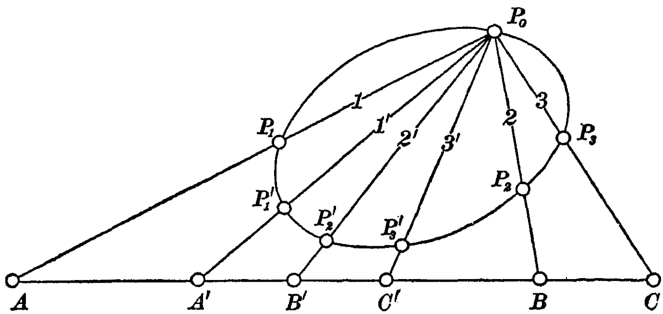


FIG. 10

terms of order relations among the points of the conic $[P]$ to mean that the same statement holds for the corresponding lines $[P_0P]$. By Theorem 6, above, together with Theorem 2, Chap. V, Vol. I, it follows that this definition is independent of the choice of the point P_0. The definitions of the order relations in the line conic, the cone of lines, and the cone of planes are made dually.*

The propositions with regard to sense are perhaps even more evident intuitionally when stated with regard to a conic or a flat pencil than with regard to the points of a line (cf. figs. 10 and 11).

* These definitions are in reality special cases of the definition given above for any one-dimensional form, since the cones and conic sections are one-dimensional forms of the second degree (§ 41, Vol. I) and since the notion of projectivity between one-dimensional forms of the first and second degrees has been defined in § 76, Vol. I. However, at present we do not need to avail ourselves of the theorems in Chap. VIII, Vol. I, on which the latter definition is based.

21. Separation of point pairs. DEFINITION. Two points A and B of a line are said to *separate* two points C and D of the same line if and only if $S(ABC) \neq S(ABD)$. This is indicated by the symbol $AB \parallel CD$.

THEOREM 7. (1) *The relation $AB \parallel CD$ implies the relations $CD \parallel AB$ and $AB \parallel DC$, and excludes the relation $AC \parallel BD$.* (2) *Given any four distinct points of a line, we have either $AB \parallel CD$ or $AC \parallel BD$* or *$AD \parallel BC$.* (3) *From the relations $AB \parallel CD$ and $AD \parallel BE$ follows the relation $AD \parallel CE$.* (4) *If $AB \parallel CD$ and $ABCD \overline{\wedge} A'B'C'D'$, then $A'B' \parallel C'D'$.**

Proof. (1) If $AB \parallel CD$, we have

$$S(ABC) \neq S(ABD), \tag{5}$$

which, by the definition of separation, implies $AB \parallel DC$. By Theorems 2–6 we obtain successively, from (5),

$$\begin{aligned} S(ABC) &= S(ADB), \\ S(ABC) &= S(ADC), \\ S(ACB) &= S(DAB), \\ S(ACB) &= S(DCB), \\ S(ABC) &= S(CDB), \\ S(CDA) &\neq S(CDB), \end{aligned}$$

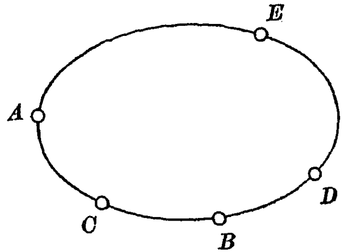


FIG. 11

the last of which implies $CD \parallel AB$. The relation $AC \parallel BD$ is excluded because it means $S(ACB) \neq S(ACD)$, which contradicts the second of the equations above.

(2) By the corollary of Theorem 3 we have either $S(ABC) \neq S(ABD)$ (in which case $AB \parallel CD$) or $S(ABC) = S(ABD)$. In the latter case either $S(ABC) \neq S(ADC)$ or $S(ABC) = S(ADC)$. The first of these alternatives is equivalent to $S(ACB) \neq S(ACD)$ and yields $AC \parallel BD$; the second implies $S(ADC) = S(ABC) = S(ABD) \neq S(ADB)$, and thus yields $AD \parallel BC$.

(3) The hypotheses give $S(ABC) \neq S(ABD)$ and $S(ADB) \neq S(ADE)$. The first of these gives $S(BCA) = S(DBA)$, which, by Theorem 5, implies $S(DBA) = S(DCA)$, and thus $S(ADB) = S(ADC)$. Hence, by the second hypothesis, $S(ADC) \neq S(ADE)$, and therefore $AD \parallel CE$.

(4) This is a direct consequence of Theorem 6.

* The properties expressed in this theorem are sufficient to define abstractly the relation of separation. Cf. Vailati, Revue de Mathématiques, Vol. V, pp. 76, 183;

THEOREM 8. *If A and B are harmonically conjugate with regard to C and D, they separate C and D.*

Proof. By Theorem 7 (2) we have either $AB \parallel CD$ or $AC \parallel BD$ or $AD \parallel BC$. We also have $ABCD \barwedge BACD$. Hence $AC \parallel BD$ would imply $BC \parallel AD$, contrary to Theorem 7 (1); and $AD \parallel BC$ would imply $BD \parallel AC$, contrary to Theorem 7 (1). Hence we must have $AB \parallel CD$.

THEOREM 9. *An involution in which two pairs separate one another has no double points.*

Proof. Suppose that the given involution had the double points M, N, and that the two pairs which separate one another are A, A' and B, B' respectively. Since the involution would be determined by the projectivity

$$MNA \barwedge MNA',$$

in which, by Theorem 8,

$$S(MNA) \neq S(MNA'),$$

it would follow, by Theorem 6, that every ordered triad was carried into an ordered triad in the opposite sense. Since the involution carries $AA'B$ to $A'AB'$, we should have

$$S(AA'B) \neq S(A'AB');$$

and hence

$$S(AA'B) = S(AA'B'),$$

contrary to hypothesis.

This theorem can also be stated in the following form:

COROLLARY 1. *An involution with double points is such that no two pairs separate one another.*

COROLLARY 2. *If an involution is direct, each pair separates every other pair. If an involution is opposite, no pair separates any other pair.*

22. Segments and intervals. DEFINITION. Let A, B, C be any three distinct points of a line. The set of all points X such that

$$S(AXC) = S(ABC)$$

is called a *segment* and is denoted by $\overline{ABC}$. The points A and C are called the *ends* of the segment. The segment $\overline{ABC}$, together with its ends, is called the *interval* ABC. The points of $\overline{ABC}$ are said to be *interior* to the interval ABC, and A and C are called its *ends.*

COROLLARY 1. *A segment does not contain its ends.*

COROLLARY 2. *If D is in $\overline{ABC}$, then*

COROLLARY 3. *If D is in $\overline{ABC}$, then B and D are not separated*
A and C.

THEOREM 10. *If A and B are any two distinct points of a line, th*
are two and only two segments, and also two and only two interva
of which A and B are ends.

Proof. Let C and D be two points which separate A and B h
monically. If X is any point of the line distinct from A and B, eith

$$S(AXB) = S(ACB)$$

or

$$S(AXB) = S(ADB).$$

In one case X is in $\overline{ACB}$, and in the other case in $\overline{ADB}$.

DEFINITION. Either of the two segments (or of the two interva
whose ends are two points A, B may be referred to as a *segment*
(or an *interval* AB). The two segments or intervals AB are said
be *complementary* to one another.

COROLLARY. *If A, B, C are any three distinct points of a line,*
line consists of the three segments complementary to $\overline{ABC}$, $\overline{BCA}$, $\overline{CA}$
together with the points A, B, and C.

Proof. Any point X distinct from A, B, C satisfies one of the re
tions $AC \parallel BX$ or $AB \parallel CX$ or $AX \parallel BC$.

THEOREM 11. *If $A_1, A_2, \cdots, A_n$ is any set of n $(n > 1)$ distinct poi*
of a line, the remaining points of the line constitute n segments, ea
of which has two of the points $A_1, A_2, \cdots, A_n$ as end points and
two of which have a point in common.

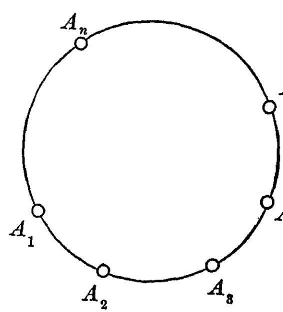


FIG. 12

Proof. The theorem is true for $n = 2$, by Theorem 10. Suppose it true for $n = k$. If $k + 1$ points are given, the point A_{k+1} is, by the theorem for the case $n = k$, on one of the k segments determined by the other k points, say on the segment whose ends are A_i and A_j. By the corollary to Theorem 10, this segment consists of A_{k+1}, together with two segments whose ends are respectively A_{k+1}, A_i and A_{k+1}, A_j. Hence the theorem is val
for $n = k + 1$ if valid for $n = k$. Hence the theorem is established
mathematical induction.

DEFINITION. A finite set of collinear points, A_i $(i = 1, \cdots, n)$, is

A set of points in the order $\{A_1A_2 \cdots A_n\}$ *is also in the orders* $\{A_2A_3 \cdots A_nA_1\}$ *and* $\{A_nA_{n-1} \cdots A_2A_1\}$.

EXERCISES

1. If $AB \parallel CD$ and $AC \parallel BE$, then $CD \parallel BE$.
2. The relations $AB \parallel CD, AB \parallel CE, AB \parallel DE$ are not possible simultaneously.
3. Any two points A, B are in the orders $\{AB\}$ and $\{BA\}$. Any three collinear points are in the orders $\{ABC\}$, $\{ACB\}$, $\{CAB\}$.

23. Linear regions. The set of all points on a line, the set of all points on a line with the exception of a single one, and the segment are examples (cf. Ex. 1 below) of what we shall define as linear regions on account of their analogy with the planar and spatial regions considered later.

DEFINITION. *A region* on a line is a set of collinear points such that (1) any two points of the set are joined by an interval consisting entirely of points of the set and (2) every point is interior to at least one segment consisting entirely of points of the set. A region is said to be *convex* if it satisfies also the condition that (3) there is at least one point of the line which is not in the set.

DEFINITION. An ordered pair of distinct points AB of a convex region R is said *to be in the same sense* as an ordered pair $A'B'$ of R if and only if $S(ABA_\infty) = S(A'B'A_\infty)$, where A_∞ is a point of the line not in R. The set of all ordered pairs of R in the same sense as AB is denoted by $S(AB)$ and is called a *sense-class.* The segment complementary to $\overline{AA_\infty B}$ is called *the segment* $\overline{AB}$. The corresponding interval is called *the interval* AB. A set of points of R is said to be in the *order* $\{A_1A_2 \cdots A_n\}$ if they are in the order $\{A_1A_2 \cdots A_nA_\infty\}$. If C is separated from A_∞ by A and B, C is *between* A and B with respect to R. If $S(AB) = S(CD)$, then C is said to *precede* D, and D to *follow* C, *in the sense* AB.

If there is a point B_∞, other than A_∞, which is not in the convex region R, the sense $S(ABA_\infty)$ is the same as the sense $S(ABB_\infty)$, and the segment $\overline{AA_\infty B}$ is the same as the segment $\overline{AB_\infty B}$. Hence

THEOREM 13. *For a given convex region* R *the above definition has the same meaning if any other point collinear with* R *but not in* R *be substituted for* A_∞.

COROLLARY 1. *If* $S(AB) = S(A'B')$ *and* $S(A'B') = S(A''B'')$, *then* $S(AB) = S(A''B'')$.

COROLLARY 2. *If* $S(AB) \neq S(A'B')$ *and* $S(A'B') \neq S(A''B'')$, *then* $S(AB) = S(A''B'')$.

COROLLARY 3. $S(AB) \neq S(BA)$.

COROLLARY 4. *If* $S(AB) = S(BC)$, *then* $S(AB) = S(AC)$.

These corollaries are direct translations of Theorems 2–5 into our present terminology. Theorem 7 translates into the following statements in terms of betweenness:

THEOREM 14. (1) *If C is between A and B, then B is not between A and C.* (2) *If three points A, B, C are distinct, C is between A and B or B is between A and C or A is between C and B.* (3) *If C is between A and B and A is between B and E, then C is between B and E.*

Theorem 7 translates into the following statements in terms of "precede" and "follows."

THEOREM 15. (1) *If C precedes B in the sense AC, then B does not precede C in this sense.* (2) *In the sense AC, either B precedes C or C precedes B.* (3) *If, in the sense AB, A precedes C and E precedes A, then E precedes C.*

DEFINITION. If A and B are any two points of a convex region R, the set consisting of all points which follow A in the sense AB is called the *ray* AB. The point A is called the *origin* of the ray. The ray consisting of all points which precede A in the sense AB is said to be *opposite* to the ray AB. The set of all points which precede A in the sense AB is sometimes called the *prolongation of the segment AB beyond A.*

EXERCISES

1. A convex region on a line is either a segment or the set of all points on the line with the exception of one point.*

2. If three points of a convex region are in the order $\{ABC\}$, they are in the order $\{CBA\}$ but not in the order $\{ACB\}$ or $\{CAB\}$.

3. In a convex region, if A is between B and C, it is between C and B.

4. Between any two points there is an infinity of points.

3. Choosing a system of nonhomogeneous coordinates in which A_∞ is ∞, show that the sense AB is the same as the sense $A'B'$ if and only if $B - A$ is of the same sign as $B' - A'$; also that two point pairs have the same sense if and only if they are conjugate under the group

$$x' = ax + b,$$

where $a > 0$.

24. Algebraic criteria of sense. If $A = (a_0, a_1)$, $B = (b_0, b_1)$, and $C = (c_0, c_1)$ are any three distinct points of the line, the transformation

$$\begin{aligned} x_0' &= \rho_0 a_0 x_0 + \rho_1 b_0 x_1, \\ x_1' &= \rho_0 a_1 x_0 + \rho_1 b_1 x_1 \end{aligned} \tag{6}$$

changes (1, 0), (0, 1), and (1, 1) into A, B, and C respectively if and only if ρ_0 and ρ_1 satisfy the equations

$$\begin{aligned} c_0 &= \rho_0 a_0 + \rho_1 b_0, \\ c_1 &= \rho_0 a_1 + \rho_1 b_1, \end{aligned}$$

that is, if

$$\frac{\rho_0}{\rho_1} = \frac{\begin{vmatrix} c_0 & b_0 \\ c_1 & b_1 \end{vmatrix}}{\begin{vmatrix} a_0 & c_0 \\ a_1 & c_1 \end{vmatrix}}.$$

With this choice of ρ_0/ρ_1 the determinant of the transformation (6) is of the same sign as

$$S = \begin{vmatrix} a_0 & b_0 \\ a_1 & b_1 \end{vmatrix} \cdot \begin{vmatrix} b_0 & c_0 \\ b_1 & c_1 \end{vmatrix} \cdot \begin{vmatrix} c_0 & a_0 \\ c_1 & a_1 \end{vmatrix}.$$

By definition the projectivity is direct if and only if S is positive. Now if $A' = (a_0', a_1')$, $B' = (b_0', b_1')$, $C' = (c_0', c_1')$ are any three points of the line, and

$$S' = \begin{vmatrix} a_0' & b_0' \\ a_1' & b_1' \end{vmatrix} \cdot \begin{vmatrix} b_0' & c_0' \\ b_1' & c_1' \end{vmatrix} \cdot \begin{vmatrix} c_0' & a_0' \\ c_1' & a_1' \end{vmatrix},$$

two cases are possible. If S' is of the same sign as S, the projectivities in which

$$(1, 0)\,(0, 1)\,(1, 1) \barwedge ABC, \tag{7}$$

$$(1, 0)\,(0, 1)\,(1, 1) \barwedge A'B'C' \tag{8}$$

are both direct or both opposite, and hence the projectivity in which

$$ABC \barwedge A'B'C' \tag{9}$$

is direct. If S' is opposite in sign to S, one of the projectivities (7) and (8) is direct and the other opposite, and hence (9) is opposite. Hence

THEOREM 16. *Let* $A = (a_0, a_1)$, $B = (b_0, b_1)$, $C = (c_0, c_1)$, $A' = (a'_0, a'_1)$, $B' = (b'_0, b'_1)$, $C' = (c'_0, c'_1)$ *be collinear points. Then* $S(ABC) = S(A'B'C')$ *if and only if the expressions*

$$\begin{vmatrix} a_0 & b_0 \\ a_1 & b_1 \end{vmatrix} \cdot \begin{vmatrix} b_0 & c_0 \\ b_1 & c_1 \end{vmatrix} \cdot \begin{vmatrix} c_0 & a_0 \\ c_1 & a_1 \end{vmatrix} \quad \textit{and} \quad \begin{vmatrix} a'_0 & b'_0 \\ a'_1 & b'_1 \end{vmatrix} \cdot \begin{vmatrix} b'_0 & c'_0 \\ b'_1 & c'_1 \end{vmatrix} \cdot \begin{vmatrix} c'_0 & a'_0 \\ c'_1 & a'_1 \end{vmatrix}$$

have the same sign.

COROLLARY 1. *Three points given by the finite nonhomogeneous coördinates a, b, c are conjugate under the group of all direct projectivities to three points given by the finite nonhomogeneous coördinates a', b', c', respectively, if and only if $(a-b)(b-c)(c-a)$ and $(a'-b')(b'-c')(c'-a')$ have the same sign.*

Proof. Set $a = a_1/a_0$, $b = b_1/b_0$, $c = c_1/c_0$, and apply the theorem.

COROLLARY 2. *Two points given by the finite nonhomogeneous coördinates a and b are conjugate under the group of all direct projectivities leaving the point ∞ of the nonhomogeneous coördinate system invariant to the two points given by the finite nonhomogeneous coördinates a' and b' respectively if and only if $a - b$ and $a' - b'$ have the same sign.*

Proof. Set $a = a_1/a_0$, $b = b_1/b_0$, $c_0 = 0$, $c_1 = 1$, and apply the theorem.

THEOREM 17. *A, B separate C, D if and only if the cross ratio* $\text{R}(AB, CD)$ *is negative.*

Proof. By the last theorem, A, B separate C, D if and only if

$$\begin{vmatrix} a_0 & b_0 \\ a_1 & b_1 \end{vmatrix} \cdot \begin{vmatrix} b_0 & c_0 \\ b_1 & c_1 \end{vmatrix} \cdot \begin{vmatrix} c_0 & a_0 \\ c_1 & a_1 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} a_0 & b_0 \\ a_1 & b_1 \end{vmatrix} \cdot \begin{vmatrix} b_0 & d_0 \\ b_1 & d_1 \end{vmatrix} \cdot \begin{vmatrix} d_0 & a_0 \\ d_1 & a_1 \end{vmatrix}$$

are opposite in sign. But the quotient of these two expressions has the same sign as $\text{R}(AB, CD)$ (cf. p. 165, Chap. VI, Vol. I).

With the aid of this theorem the proof of Theorem 7 can be made much more simply than in § 21.

25. Pairs of lines and of planes. THEOREM 18. *The points of space not on either of two planes α and β fall into two classes such that two points O_1, O_2 of the same class are not separated by the points in which the line O_1O_2 meets the planes α and β, while two points O, P of different classes are separated by the points in which the line OP meets α and β.*

Proof. By the space dual of Theorem 10 the planes of the pencil $\alpha\beta$ are separated by α and β into two segments. Let $[O]$ be the set

segment but not on the line $\alpha\beta$.

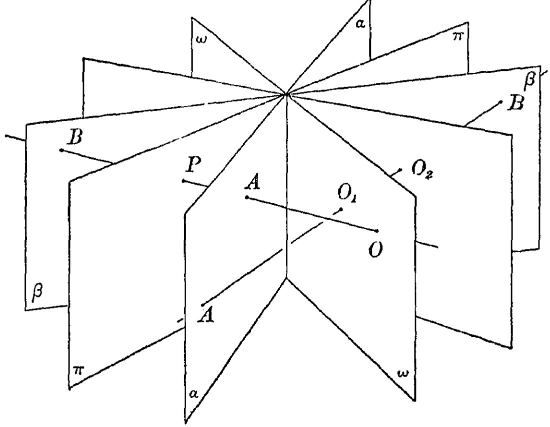


FIG. 13

The two planes ω and π of the pencil $\alpha\beta$ which are on any two points O and P are separated by α and β. Hence, by Theorem 7 and § 20, the points in which the line OP meets α and β are separated by O and P. In like manner, any two points O_1, O_2 determine with the line $\alpha\beta$ a pair of planes (or a single plane) not separated by α and β, and hence the line O_1O_2 meets α and β in points (or a single point) not separated by O_1 and O_2. By the same reasoning, any line P_1P_2 meets α and β in points (or a point) not separated by P_1 and P_2.

COROLLARY 1. *If l and m are two coplanar lines, the points of the plane which are not on l or m fall into two classes such that two points O_1, O_2 of the same class are not separated by the points in which the line O_1O_2 meets l and m, while two points O, P of different classes are separated by the points in which OP meets l and m.*

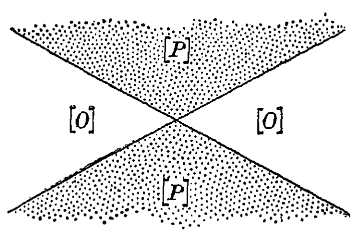


FIG. 14

COROLLARY 2. *There is only one pair of classes [O] and [P] satisfying the conditions of the above theorem (or its first corollary) determined by a given pair of planes (or lines).*

DEFINITION. Two points in different classes (according to Corollary 1) relative to two coplanar lines are said to be *separated* by the two lines; otherwise they are said not to be separated by the lines. Two points in different classes (according to Theorem 18) relative to two planes are said to be *separated* by the two planes; otherwise they are said not to be separated by the planes.

1. If l_1 and l_2 are two coplanar lines and O any point of their common plane, all triads of points in a fixed sense-class S_1 on l_1 are projected from O into triads in a fixed sense-class S_2 on l_2 (Theorem 6). If P is any other point of the plane, it is separated from O by l_1 and l_2 if and only if triads in the sense S_1 are not projected from P into triads in the sense S_2.

This problem can be stated also in terms of the sense of pairs of points in the region obtained on l_1 or l_2 respectively by leaving out the common point. The theorem in this form is generalized in § 30. In the form stated in Ex. 1 it has the following generalization.

2. If l_1 and l_2 are two noncoplanar lines, and o is any line not intersecting them, all triads in a fixed sense S_1 on l_1 are axially projected from o into triads in a fixed sense S_2 on l_2 (Theorem 6). The lines not intersecting l_1 and l_2 fall into two classes: those by which triads in the sense S_1 are projected into triads in the sense S_2, and those by which triads in the sense S_1 are projected into triads in the sense opposite to S_2.

3. Obtain the definition of separation of two coplanar lines by two points as the plane dual of the definition of separation of two points by two coplanar lines. Prove that if two coplanar lines separate two points, then the points separate the lines. State and prove the corresponding result for pairs of points and of planes.

26. The triangle and the tetrahedron.

THEOREM 19. *If a line l not passing through any vertex of a triangle ABC meets the sides BC, CA, AB in A_1, B_1, C_1 respectively, then any other line m which meets the segments $\overline{BA_1C}$, $\overline{CB_1A}$ also meets the segment $\overline{AC_1B}$.*

Proof. Suppose first that m passes through A_1; then

$$ACB_1B_2 \overset{A_1}{\underset{\wedge}{=}} (ABC_1C_2),$$

and hence, if B_1 and B_2 do not separate A and C, C_1 and C_2 do not separate A and B. Similarly, the theorem is true if m passes through B_1.

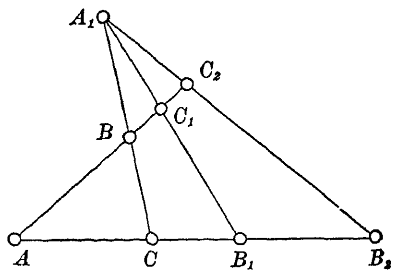


FIG. 15

If m does not pass through A_1 or B_1, let m' be a line joining A_1 to the point in which m meets CA. By the argument above we have first that m' meets all three segments $\overline{BA_1C}$, $\overline{CB_1A}$, and $\overline{AC_1B}$, and then that m meets them.

Let us denote the segment $\overline{AC_1B}$ by γ, $\overline{BA_1C}$ by α, and $\overline{CB_1A}$ by β,

above theorem then gives the information that every line which meets two of the segments α, β, γ meets the third. Any line which meets α and $\bar{\beta}$ meets $\bar{\gamma}$, for, as it does not pass through A or B, it meets either γ or $\bar{\gamma}$; but if it met γ, and by hypothesis meets α, it would meet β. Hence the theorem gives that α, $\bar{\beta}$, $\bar{\gamma}$ are such that any line meeting two of these segments meets the third. By a repetition of this argument it follows that every line of the plane which does not pass through a vertex of the triangle meets all three segments of one of the trios $\alpha\beta\gamma$, $\bar{\alpha}\bar{\beta}\gamma$, $\bar{\alpha}\beta\bar{\gamma}$, $\alpha\bar{\beta}\bar{\gamma}$, and no line whatever meets all three segments in any of the trios $\alpha\beta\bar{\gamma}$, $\alpha\bar{\beta}\gamma$, $\bar{\alpha}\beta\gamma$, $\bar{\alpha}\bar{\beta}\bar{\gamma}$.

The lines of the plane, exclusive of those through the vertices, therefore fall into four classes:

(1) those which meet α, β, γ,
(2) those which meet $\bar{\alpha}$, $\bar{\beta}$, γ,
(3) those which meet α, $\bar{\beta}$, $\bar{\gamma}$,
(4) those which meet $\bar{\alpha}$, β, $\bar{\gamma}$.

No two lines l_1, l_2 of the same class are separated by any pair of the lines joining the point l_1l_2 to the vertices of the triangle, while any two lines l_1, m_1 of different classes are separated by two of the lines joining the point l_1m_1 to the vertices. This result is perhaps more intuitively striking when put into the dual form, as follows:

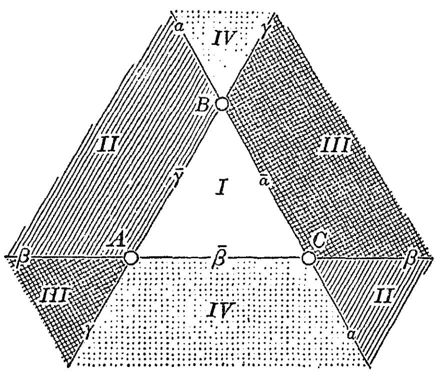


FIG. 16

THEOREM 20. *The points of a plane not on the sides of a triangle fall into four classes such that no two points L_1, L_2 of the same class are separated by any pair of the points in which the line L_1L_2 meets the sides of the triangle, while any two points L_1, M_1 of different classes are separated by two of the points in which the line L_1M_1 meets the sides of the triangle.*

DEFINITION. Any one of the four classes of points in Theorem 20 is called a *triangular region*. The vertices of the triangle are also called *vertices of the triangular region*.

The property of the triangle stated in Theorem 19 can also ser as a basis for a discussion of the ordinal theorems on the tetrahedr and for those of the $(n+1)$-point in n-space. Suppose we have a tetr hedron whose vertices are A_1, A_2, A_3, A_4. Let us denote its faces by $\alpha_2, \alpha_3, \alpha_4$, the face α_1 being opposite to the vertex A_1, etc.; let us denc the edges by $a_{12}, a_{13}, a_{14}, a_{23}, a_{34}, a_{42}$, the edge a_{ij} being the line A_i Each edge a_{ij} is separated by the vertices A_i, A_j into two segmen which we shall denote by σ_{ij} and $\bar{\sigma}_{ij}$. Let π be a plane not passi through any vertex; the six segments which it meets may be denot by $\sigma_{12}, \sigma_{13}, \cdots, \sigma_{42}$, and the complementary segments by $\bar{\sigma}_{12}, \bar{\sigma}_{13}, \cdots, \bar{\sigma}$ Then as a corollary of Theorem 19 we have that any plane which meets three noncoplanar segments of the set $\sigma_{12}, \sigma_{13}, \cdots, \sigma_{42}$ meets all the rest of them, and, moreover, no plane meets all the segments $\bar{\sigma}_{12}, \bar{\sigma}_{13}, \cdots, \bar{\sigma}_{42}$. If we observe that any plane not passing through a vertex must meet the edges a_{12}, a_{13}, a_{14} in three distinct points, it becomes clear that the planes n passing through any vertex fall into eight classes such that two plan of the same class are not separated by a pair of vertices, where two planes of different classes are separated by a pair of vertice Under duality we have

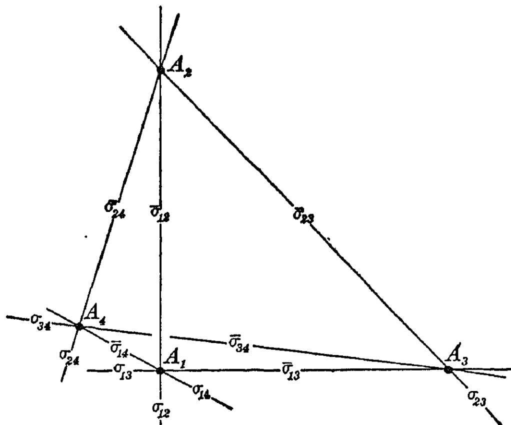


FIG. 17

THEOREM 21. *The points not upon the faces of a tetrahedron fa into eight classes such that two points of the same class are not sep rated by the points in which the line joining them meets the face whereas two points of different classes are separated by two of t points in which their line meets the faces of the tetrahedron.*

DEFINITION. Any one of the eight classes of points in Theorem 2 is called a *tetrahedral region.* The vertices of the tetrahedron a

and to prove that the boundary of any one of the classes of points in Theorem 20 is composed of A, B, C and three segments having the property that no line meets them all. We shall defer this discussion, however, to a later chapter, where the results will appear as special cases of more general theorems.

27. Algebraic criteria of separation. Cross ratios of points in space. The classes of points determined (Theorems 18–21) by a pair of intersecting lines, a triangle, a pair of planes or by a tetrahedron can be discussed by means of some very elementary algebraic considerations. As these are similar in the plane and in space, let us carry out the work only for the three-dimensional cases.

Suppose that the homogeneous coördinates of four noncoplanar points A_1, A_2, A_3, A_4 are given by the columns of the matrix,

$$(10) \qquad \begin{pmatrix} a_{01} & a_{02} & a_{03} & a_{04} \\ a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{pmatrix},$$

and let (x_0, x_1, x_2, x_3) be the homogeneous coördinates of any other point X. Let us indicate by $|x, a_2, a_3, a_4|$ the determinant of the matrix obtained by substituting x_0, x_1, x_2, x_3 respectively for the elements of the first column in the matrix above; by $|a_1, x, a_3, a_4|$ the determinant obtained by performing the same operation on the second column, etc. The expressions $|y, a_2, a_3, a_4|$ etc. have similar meanings in terms of the coördinates of a point $(y_0, y_1, y_2, y_3) = Y$. The following expressions are formed analogously to the cross ratios of four points on a line (cf. § 58, Vol. I):

$$(11) \qquad \begin{aligned} k_{14} &= \frac{|x, a_2, a_3, a_4|}{|a_1, a_2, a_3, x|} \div \frac{|y, a_2, a_3, a_4|}{|a_1, a_2, a_3, y|}, \\ k_{24} &= \frac{|a_1, x, a_3, a_4|}{|a_1, a_2, a_3, x|} \div \frac{|a_1, y, a_3, a_4|}{|a_1, a_2, a_3, y|}, \\ k_{34} &= \frac{|a_1, a_2, x, a_4|}{|a_1, a_2, a_3, x|} \div \frac{|a_1, a_2, y, a_4|}{|a_1, a_2, a_3, y|}. \end{aligned}$$

Clearly there are twelve numbers k_{ij} which could be defined analogously to these; and if the notation A_1, A_2, A_3, A_4, X, Y be permuted among the six points, 720 such expressions are defined. Each number k_{ij}

coördinates of any point be multiplied by a constant or if all six points be subjected to the same linear transformation.

If Y be not upon any of the planes determined by the points A_1, A_2, A_3, A_4, there exists a projectivity which carries Y into $(1, 1, 1, 1)$ and the points A_1, A_2, A_3, A_4 into the points represented by the columns of

$$(12) \qquad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Let (X_0, X_1, X_2, X_3) be the point into which (x_0, x_1, x_2, x_3) is carried by this projectivity. By substituting in (11) we see that

$$k_{14} = \frac{X_0}{X_3}, \quad k_{24} = \frac{X_1}{X_3}, \quad k_{34} = \frac{X_2}{X_3}.$$

From this it follows that $|x, a_2, a_3, a_4|$, $|a_1, x, a_3, a_4|$, etc. *could be taken as the homogeneous coördinates with respect to the tetrahedron of reference whose vertices are* A_1, A_2, A_3, A_4.

The line $\qquad (X_0, X_1, X_2, X_3) - \lambda(1, 1, 1, 1)$

meets the planes determined by the four points represented by (12) in four points given by the values $\lambda = X_0, \lambda = X_1, \lambda = X_2, \lambda = X_3$. The cross ratios of pairs of these points with (X_0, X_1, X_2, X_3) and $(1, 1, 1, 1)$ are X_0/X_3, X_1/X_3, and X_2/X_3. Hence k_{14}, k_{24}, k_{34} *are cross ratios of X and Y with pairs of points in which the line joining them meets the faces of the tetrahedron* $A_1A_2A_3A_4$.

By Theorem 17, the points X and Y are separated by the planes $A_2A_3A_4$ and $A_2A_3A_1$ if and only if k_{14} is negative. They will be separated by $A_1A_3A_4$ and $A_1A_3A_2$ if and only if k_{24} is negative, and by $A_1A_2A_4$ and $A_1A_2A_3$ if and only if k_{34} is negative. Hence, by Theorem 21, we have

THEOREM 22. *The points X and Y will be in the same class with respect to the tetrahedron* $A_1A_2A_3A_4$ *if and only if* k_{14}, k_{24}, k_{34} *are all positive.*

COROLLARY. *The eight regions determined by the tetrahedron* $A_1A_2A_3A_4$ *are those for which the algebraic signs of* k_{14}, k_{24}, k_{34} *appear in the following combinations:* $(+, +, +)$, $(+, +, -)$, $(+, -, +)$, $(-, +, +)$, $(-, -, -)$, $(-, -, +)$, $(-, +, -)$, $(+, -, -)$.

Recalling that $|x, a_2, a_3, a_4| = 0$ is the equation of the plane $A_2A_3A_4$ (cf. § 70, Vol. I), we see that if

$$\alpha(x) \equiv \alpha_0 x_0 + \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = 0$$

and

$$\beta(x) \equiv \beta_0 x_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 = 0$$

are the equations of two planes, the formula given above for the cross ratio of two points X and Y with the points of intersection of the line XY with these planes becomes

$$\frac{\alpha(x)}{\alpha(y)} \div \frac{\beta(x)}{\beta(y)}. \tag{13}$$

Thus two points are in the same one of the two classes determined by the planes $\alpha(x)$ and $\beta(x)$ if and only if this expression is positive.

This result assumes an even simpler form when specialized somewhat with respect to a system of nonhomogeneous coördinates. Suppose that $x_0 = 0$ be chosen as the singular plane in a system of nonhomogeneous coördinates; then the same point is represented nonhomogeneously by (x, y, z) or homogeneously by $(1, x, y, z)$, and the plane represented above by $\alpha(x) = 0$ has the equation

$$\alpha_1 x + \alpha_2 y + \alpha_3 z + \alpha_0 = 0.$$

If $\beta(x) = 0$ be the plane $x_0 = 0$, the expression for the cross ratio written above becomes

$$\frac{\alpha(x)}{\alpha(y)} \div \frac{x_0}{y_0},$$

which reduces in nonhomogeneous coördinates, when (x_0, x_1, x_2, x_3) and (y_0, y_1, y_2, y_3) are replaced by $(1, x', y', z')$ and $(1, x'', y'', z'')$, to

$$\frac{\alpha_1 x' + \alpha_2 y' + \alpha_3 z' + \alpha_0}{\alpha_1 x'' + \alpha_2 y'' + \alpha_3 z'' + \alpha_0}. \tag{14}$$

Hence two points (x', y', z') and (x'', y'', z'') are separated by the singular plane, and $\alpha_1 x + \alpha_2 y + \alpha_3 z + \alpha_0 = 0$ if and only if the numerator and denominator of (14) are of opposite sign. For reference we shall state this as a theorem in the following form:

THEOREM 23. *The two classes of points determined, according to Theorem 18, by the singular plane of a nonhomogeneous coördinate system and a plane $ax + by + cz + d = 0$ are respectively the points*

EXERCISES

1. Carry out the discussion analogous to the above in the two-dimensio case. Generalize to n dimensions.

2. How many of the 720 numbers analogous to k_{14} are distinct?

28. Euclidean spaces. DEFINITION. The set of all points of a p jective space* of n dimensions, with the exception of those on a sin $(n-1)$-space S^∞ contained in the n-space, is called a *Euclidean sp of n dimensions.* Thus, in particular, the set of all but one of t points of a projective line is called a *Euclidean line,* and the set all the points of a projective plane, except those on a single line, called a *Euclidean plane.*

DEFINITION. The projective $(n-1)$-space S^∞ is called the *singul $(n-1)$-space* or *the $(n-1)$-space at infinity* or the *ideal $(n-1)$-sp associated with* the Euclidean space. Any figure in S^∞ is said to *ideal* or to be *at infinity,* whereas any figure in the Euclidean n-spa is said to be *ordinary.*

The ordinary points of any line in a Euclidean plane or space for a Euclidean line and thus satisfy the definition (§ 23) of a line convex region. The definitions and theorems of that section m therefore be applied at once in discussing Euclidean spaces. Th if A and B are any two ordinary points, we shall speak of "the se ment AB," "the ray AB," etc.

The first corollary of Theorem 18 yields a very simple and imp tant theorem if the line m be taken as the line at infinity, namely

THEOREM 24. *The points of a Euclidean plane which are not on line l fall into two classes such that the segment joining two points the same class does not meet l and the segment joining two poi of different classes does meet l.*

COROLLARY. *If a is any ray whose origin is a point of l, all poi of a are either on l or on the same side of l.*

In like manner Theorem 18 yields

THEOREM 25. *The points of a Euclidean three-space which are not a plane π fall into two classes such that the segment joining two poi of the same class does not meet π and the segment joining two poi of different classes does meet π.*

* We shall refer to a line, plane, or n-space in the sense of Chap. I, Vol. I, a projective line, plane, or n-space whenever there is possibility of confusion w

DEFINITION. The two classes of points determined by a line l in a Euclidean plane, according to Theorem 24, are called the two *sides* of l. The two classes of points determined by a plane π in a Euclidean three-space, according to Theorem 25, are called the two *sides* of π.

The two sides of π are characterized algebraically in Theorem 23.

DEFINITION. An ordered pair of rays h, k having a common origin is called an *angle* and is denoted by $\measuredangle hk$. If the rays are AB and AC, the angle may also be denoted by $\measuredangle BAC$. If the rays are opposite, the angle is called a *straight* angle; if the rays coincide, it is called a *zero* angle. The rays h, k are called the *sides* of $\measuredangle hk$, and their common origin the *vertex* of $\measuredangle hk$.

EXERCISES

1. The points of a Euclidean plane not on the sides or vertex of a nonzero angle $\measuredangle hk$ fall into two classes such that the segment joining two points of different classes contains one point of h or k. In case $\measuredangle hk$ is not a straight angle, one of these two classes consists of every point which is between a point of h and a point of k.

2. Generalize Theorem 25 to n dimensions.

29. Assumptions for a Euclidean space. A Euclidean space can be characterized completely by means of a set of assumptions stated in terms of order relations. Such a set of assumptions is given below. It is a simple exercise, which we shall leave to the reader, to verify that these assumptions are all satisfied by a Euclidean space as defined in the last section.

The reverse process is also of considerable interest. This consists (1) in deriving the elementary theorems of alignment and order from Assumptions I–VIII below, and (2) in defining ideal elements and showing that these, together with the elements of the Euclidean space, form a projective space. For the details of (1) and an outline of (2) the reader may consult the article by the writer, in the Transactions of the American Mathematical Society, Vol. V (1904), pp. 343–384, and also a note by R. L. Moore, in the same journal, Vol. XIII (1912), p. 74. On (2) one may consult the article by R. Bonola, Giornale di Matematiche, Vol. XXXVIII (1900), p. 105, and also that by F. W. Owens, Transactions of the American Mathematical Society, Vol. XI (1910), p. 141. Compare also the Introduction to Vol. I.

This set of assumptions refers to an undefined class of elements called points and an undefined relation among points indicated by saying "the points A, B, C are in the order $\{ABC\}$."

The assumptions are as follows:

I. *If points A, B, C are in the order $\{ABC\}$, they are distinct.*

II. *If points A, B, C are in the order $\{ABC\}$, they are not in the order $\{BCA\}$.*

DEFINITION. If A and B are distinct points, the *segment* $\overline{AB}$ consists of all points X in the order $\{AXB\}$; all points of the segment $\overline{AB}$ are said to be *between* A and B; the segment together with A and B is called the *interval* AB; the *line* AB consists of A and B and all points X in one of the orders $\{ABX\}$, $\{AXB\}$, $\{XAB\}$; and the *ray* AB consists of B and all points X in one of the orders $\{AXB\}$ and $\{ABX\}$.

III. *If points C and D $(C \neq D)$ are on the line AB, then A is on the line CD.*

IV. *If three distinct points A, B, and C do not lie on the same line, and D and E are two points in the orders $\{BCD\}$ and $\{CEA\}$, then a point F exists in the order $\{AFB\}$ and such that D, E, and F lie on the same line.*

V. *If A and B are two distinct points, there exists a point C such that A, B, and C are in the order $\{ABC\}$.*

VI. *There exist three distinct points A, B, C not in any of the orders $\{ABC\}$, $\{BCA\}$, $\{CAB\}$.*

DEFINITION. If A, B, C are three noncollinear points, the set of all points collinear with pairs of points on the intervals AB, BC, CA is called the *plane* ABC.

VII. *If A, B, C are three noncollinear points, there exists a point D not in the same plane with A, B, and C.*

VIII. *Two planes which have one point in common have two distinct points in common.*

IX. *If A is any point and a any line not containing A, there is not more than one line through A coplanar with a and not meeting a.*

XVII. *If there exists an infinitude of points, there exists a certain pair of points A, C such that if $[\sigma]$ is any infinite set of segments of the line AC, having the property that each point of the interval AC is a point of a segment σ, then there is a finite subset, $\sigma_1, \sigma_2, \cdots, \sigma_n$, with the same property.**

* The proposition here stated about the interval AC is commonly known as the Heine-Borel theorem. The continuity assumption is more usually stated in the form of the "Dedekind Cut Axiom." Cf. R. Dedekind, Stetigkeit und irrationalen

Assumptions I to VIII are sufficient to define a three-space which is capable of being extended by means of ideal elements into a projective space satisfying A, E, S. This space will not, in general, satisfy Assumption P. If the continuity assumption, XVII, be added, the corresponding projective space is real and hence properly projective. Assumption IX is the assumption with regard to parallel lines. Assumption VIII limits the number of dimensions to three.

30. Sense in a Euclidean plane. Suppose that l_∞ is the line at infinity of a Euclidean plane. Every collineation transforming the Euclidean plane into itself effects a projectivity on l_∞ which is either direct or opposite (§ 18). Since the direct projectivities on l_∞ form a group, the planar collineations which effect these transformations on l_∞ also form a group.

DEFINITION. A collineation of a Euclidean plane which effects a direct projectivity on the line at infinity of this plane is said to be a *direct collineation* of the Euclidean plane. Any other collineation of the Euclidean plane is said to be *opposite*. Let A, B, C be three noncollinear points; the class of all ordered triads $A'B'C'$ such that the collineation carrying A, B, and C to A', B', and C' respectively is direct, is called a *sense-class* and is denoted by $S(ABC)$. Two ordered triads of noncollinear points in the same sense-class are said to *have the same sense* or to *be in the same sense*. Otherwise they are said to *have opposite senses* or to *be in opposite senses.*

Since the direct projectivities form a group, it follows that if a triad $A'B'C'$ is in $S(ABC)$, then $S(ABC) = S(A'B'C')$.

THEOREM 26. *There are two and only two sense-classes in a Euclidean plane. If A, B, and C are noncollinear points,* $S(ABC) = S(BCA) \neq S(ACB)$.

Proof. Let A, B, C be three noncollinear points. If A', B', C' are any three noncollinear points such that the projectivity carrying A, B, C to A', B', C' respectively is direct, $S(ABC)$ contains the triad $A'B'C'$. Because the direct projectivities form a group, $S(ABC) = S(A'B'C')$. The triads to which ABC is carried by collineations which are not direct all form a sense-class, because the product of two opposite collineations is direct. Thus there are two and only two sense-classes.

Suppose we denote the lines BC, CA, AB by a, b, c respectively and let A', B', C' be the points of intersection of a, b, c respectively

with l_∞. The projectivity carrying ABC to BCA evidently ca
a, b, and c to b, c, and a respectively, and thus carries $A'B'C'$ to $B'C$
and thus is direct (§ 19). Hence

$$S(ABC) = S(BCA).$$

The projectivity carrying ABC to ACB carries $A'B'C'$ to $A'C'B'$,
hence is not direct; and hence

$$S(ABC) \neq S(ACB).$$

THEOREM 27. *Two points C and D are on opposite sides of a
AB if and only if* $$S(ABC) \neq S(ABD).$$

This theorem can be derived as a consequence of Ex. 1, § 25
can also be derived from the following algebraic considerations.

Let us choose a system of nonhomogeneous coördinates in su
way that the singular line of the coördinate system is the sam
the singular line of the Euclidean plane. The group of all pr
tive collineations transforming the Euclidean plane into itself
reduces (§ 67, Vol. I) to

$$(15)\qquad \begin{aligned} x' &= a_1x + b_1y + c_1, \\ y' &= a_2x + b_2y + c_2, \end{aligned} \qquad \Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0.$$

If we change to the homogeneous coördinates for which $x =$
and $y = x_2/x_0$, the line at infinity has the equation $x_0 = 0$, and
equations (15) reduce to

$$(16)\qquad \begin{aligned} x_0' &= x_0, \\ x_1' &= c_1x_0 + a_1x_1 + b_1x_2, \\ x_2' &= c_2x_0 + a_2x_2 + b_2x_2. \end{aligned}$$

On the line at infinity this effects the transformation

$$\begin{aligned} x_1' &= a_1x_1 + b_1x_2, \\ x_2' &= a_2x_1 + b_2x_2, \end{aligned}$$

which is direct if and only if $\Delta > 0$.

Let the nonhomogeneous coördinates of three points A, B,
(a_1, a_2), (b_1, b_2), (c_1, c_2) respectively. The determinant

$$(17)\qquad S = \begin{vmatrix} a_1 & a_2 & 1 \\ b_1 & b_2 & 1 \\ c_1 & c_2 & 1 \end{vmatrix}$$

is multiplied by Δ whenever the points A, B, C are subjected to
transformation (15). This is verified by a direct substitution. H

changed by all others. Hence we have

THEOREM 28. *An ordered triad of points* (a_1, a_2), (b_1, b_2), (c_1, c_2) *has the same sense as an ordered triad* (a_1', a_2'), (b_1', b_2'), (c_1', c_2') *if and only if the determinants*

$$\begin{vmatrix} a_1 & a_2 & 1 \\ b_1 & b_2 & 1 \\ c_1 & c_2 & 1 \end{vmatrix} \quad \textit{and} \quad \begin{vmatrix} a_1' & a_2' & 1 \\ b_1' & b_2' & 1 \\ c_1' & c_2' & 1 \end{vmatrix}$$

have the same sign.

Theorem 27 now follows as a corollary of Theorem 23, § 27.

EXERCISES

1. If $\measuredangle ABC = \measuredangle A'BC'$, $S(ABC) = S(A'BC')$.

2. Let $\measuredangle hk$ be said to have the same sense as $\measuredangle h'k'$ if $S(ABC) = S(A'B'C')$, where B is the vertex of $\measuredangle hk$, A a point of h, C a point of k, and A', B', C' points analogously defined for $\measuredangle h'k'$. Define positive and negative angles and develop a theory of the order relations of rays through a point.

3. Let ρ and σ be two planes of a projective space which meet in a line l_∞; let us denote the two Euclidean planes obtained by leaving l_∞ out of ρ and σ by ρ_1 and σ_1 respectively; and let S_ρ be an arbitrary sense-class in ρ_1. All ordered point triads of S_ρ are projected from a point O not on ρ or σ into triads of a fixed sense-class S_σ in σ_1. Any other point P not on ρ or σ is separated from O by ρ and σ if and only if triads in the sense-class S_ρ are not projected from P into triads of S_σ.

***31. Sense in Euclidean spaces.** The definition given above of direct transformations in a Euclidean plane, based on the concept of direct transformations on the singular line, cannot be generalized to three dimensions. This is because the plane at infinity is projective and, as will be proved in the next section, does not admit of a distinction between direct and opposite projectivities. Nevertheless, the algebraic criterion $\Delta > 0$ does generalize and is made the basis of the definition which follows.

With reference to a nonhomogeneous coördinate system, of which the singular $(n-1)$-space is the $(n-1)$-space at infinity, the equations of any projective collineation of a Euclidean n-space take the form*

$$x_i' = b_i + \sum_{j=1}^{n} a_{ij} x_j, \qquad (i = 1, \cdots, n) \tag{18}$$

where the determinant $|a_{ij}|$ is different from zero. The resultant of

* The reader may, if he wishes, limit attention to the case $n = 3$. We have not actually developed the theory of coördinate systems in n dimensions, but as there is no essential difference in this theory between the three-dimensional case and the n-dimensional, we do not intend to write out the details.

two transformations of this form has a determinant which is the product of the determinants of the two transformations. Since the coefficients appear nonhomogeneously in (18), it is clear that a self-conjugate subgroup of the group of all transformations (18) is defined by the condition $|a_{ij}| > 0$. It follows by the same reasoning as used in § 18 that this subgroup is independent of the choice of the frame of reference, so long as the singular $(n-1)$-space coincides with the singular $(n-1)$-space of the corresponding Euclidean n-space.

DEFINITION. The group of all transformations (18) for which the determinant $|a_{ij}| > 0$ is called the group of *direct* collineations. In a Euclidean n-space let $A_1, A_2, \cdots, A_{n+1}$ be $n+1$ linearly independent points; the class of all ordered $(n+1)$-ads* $A_1'A_2' \cdots A_{n+1}'$ such that the collineation transforming $A_1, A_2, \cdots, A_{n+1}$ into $A_1', A_2', \cdots, A_{n+1}'$ respectively is direct is called a *sense-class* and is denoted by $S(A_1A_2 \cdots A_{n+1})$.

THEOREM 29. *There are two and only two sense-classes in a Euclidean n-space. The sense-class of an ordered n-ad is unaltered by even permutations and altered by odd permutations.*

Proof. The argument for the three-dimensional case is typical of the general case. Let the coördinates of four points A, B, C, D be (a_1, a_2, a_3), (b_1, b_2, b_3), (c_1, c_2, c_3), (d_1, d_2, d_3) respectively. The determinant

$$(19) \qquad \begin{vmatrix} a_1 & a_2 & a_3 & 1 \\ b_1 & b_2 & b_3 & 1 \\ c_1 & c_2 & c_3 & 1 \\ d_1 & d_2 & d_3 & 1 \end{vmatrix}$$

is multiplied by $|a_{ij}|$ whenever the points are simultaneously subjected to a transformation (18). Hence the algebraic sign of (19) is left invariant by all direct collineations.

Since an odd permutation of the rows of (19) would change the sign of (19), no such permutation can be effected by a direct collineation. The remaining statements in the theorem now follow directly from the theorem that any ordered tetrad of points can be transformed by a transformation of the form (18) into any other ordered tetrad.

***32. Sense in a projective space.** Let us consider the group of all linear transformations

$$(20) \qquad x_i' = \sum_{j=0}^{n} a_{ij}x_j, \qquad (i = 0, \cdots, n)$$

for which the determinant $|a_{ij}|$ is different from zero.

If $(x_0, \cdots, x_n)$ is a set of homogeneous coördinates, the equations (20) continue to represent the same transformation when all the a_{ij}'s are multiplied by the same constant ρ; and two sets of equations like (20) represent the same transformation only if the coefficients of one are proportional to those of the other.

If each a_{ij} be multiplied by ρ, $|a_{ij}|$ is multiplied by ρ^{n+1}. Hence, if $|a_{ij}|$ is negative *and n is even*, we may multiply each a_{ij} by -1 and thus obtain an equivalent expression of the form (20) for which $|a_{ij}|$ is positive. If, however, n is odd, $\rho^{n+1} = k < 0$ has no real root. Hence, if n is odd, a transformation (20) for which $|a_{ij}|$ is negative is not equivalent to one for which $|a_{ij}|$ is positive. Hence the condition $|a_{ij}| > 0$ determines a subset of the transformations (20) if and only if n is odd. This subset of transformations forms a group for the reason given in § 18 for the case $n = 1$.

DEFINITION. If n is odd, the group of transformations (20) for which $|a_{ij}| > 0$ is called the group of *direct* collineations in n-space.

This definition of the group of direct collineations is independent of the choice of the frame of reference, as follows by an argument precisely like that used to prove the corresponding proposition in § 18.

In a space of three dimensions, let us inquire into what sets of five points the set (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1), (1, 1, 1, 1) can be transformed by direct collineations. If the initial points are to be transformed respectively into the points whose coördinates are the columns of the matrix

$$(21)\qquad \begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} & a_{04} \\ a_{10} & a_{11} & a_{12} & a_{13} & a_{14} \\ a_{20} & a_{21} & a_{22} & a_{23} & a_{24} \\ a_{30} & a_{31} & a_{32} & a_{33} & a_{34} \end{pmatrix},$$

the collineation must take the form

$$(22)\qquad \begin{aligned} x_0' &= \rho_0 a_{00} x_0 + \rho_1 a_{01} x_1 + \rho_2 a_{02} x_2 + \rho_3 a_{03} x_3, \\ x_1' &= \rho_0 a_{10} x_0 + \rho_1 a_{11} x_1 + \rho_2 a_{12} x_2 + \rho_3 a_{13} x_3, \\ x_2' &= \rho_0 a_{20} x_0 + \rho_1 a_{21} x_1 + \rho_2 a_{22} x_2 + \rho_3 a_{23} x_3, \\ x_3' &= \rho_0 a_{30} x_0 + \rho_1 a_{31} x_1 + \rho_2 a_{32} x_2 + \rho_3 a_{33} x_3, \end{aligned}$$

where the ρ's satisfy the equations

$$(23)\qquad \begin{aligned} \rho_0 a_{00} + \rho_1 a_{01} + \rho_2 a_{02} + \rho_3 a_{03} &= a_{04}, \\ \rho_0 a_{10} + \rho_1 a_{11} + \rho_2 a_{12} + \rho_3 a_{13} &= a_{14}, \\ \rho_0 a_{20} + \rho_1 a_{21} + \rho_2 a_{22} + \rho_3 a_{23} &= a_{24}, \\ \rho_0 a_{30} + \rho_1 a_{31} + \rho_2 a_{32} + \rho_3 a_{33} &= a_{34}, \end{aligned}$$

this determinant is

$$(24) \qquad \frac{(a_{04}a_{11}a_{22}a_{33})(a_{00}a_{14}a_{22}a_{33})(a_{00}a_{11}a_{24}a_{33})(a_{00}a_{11}a_{22}a_{34})}{(a_{00}a_{11}a_{22}a_{33})^3},$$

where the expressions in parentheses are abbreviations for determinants formed from the matrix (21) having these expressions as their main diagonals. The number (24) has the same sign as

$$(25) \quad (a_{04}a_{11}a_{22}a_{33})(a_{00}a_{14}a_{22}a_{33})(a_{00}a_{11}a_{24}a_{33})(a_{00}a_{11}a_{22}a_{34})(a_{00}a_{11}a_{22}a_{33}),$$

which is entirely analogous to the expression found in Theorem 16. The initial set of points is transformable into the points whose coördinates are the columns of (21) by a direct transformation if and only if (25) is positive.

This result may be stated in the form of a theorem as follows:

THEOREM 30. *If a set of five points whose homogeneous coördinates are the columns of the matrix* (21) *be such that the product of the four-rowed determinants obtained by omitting columns of this matrix is positive, it can be transformed by a direct collineation into any other set of points having the same property, but not into a set for which the analogous product is zero or negative.*

COROLLARY. *Any even permutation but no odd permutation of the vertices of a complete five-point can be effected by a direct collineation.*

DEFINITION. Let A, B, C, D, E be five points no four of which are coplanar. The class of all ordered pentads obtainable from the pentad A, B, C, D, E by direct collineations is called a *sense-class* and is denoted by $S(ABCDE)$.

Theorem 30 and its corollary now give at once the following:

THEOREM 31. *There are two and only two sense-classes in a real projective three-space. The sense-class of a set of five points is unaltered by even permutations and altered by odd permutations.*

If an analogous definition of sense-class had been made in the plane, we should have had that all planar collineations are direct, and hence that there is only one sense-class in the plane. This remark, together with Theorem 31, expresses in part what is meant by the proposition:

The real projective plane is one-sided and the real projective three-space is two-sided.

Although we have grounded this discussion upon propositions regarding certain groups of collineations, the notion of sense is connected with a much more extensive group. We shall return to this study, which will give a deeper insight into the notions of sense and of one- and two-sidedness, in a later chapter.

33. Intuitional description of the projective plane. We may assist our intuitive conception* of the one-sidedness of the real projective plane by a further consideration of the regions into which a plane is separated by a triangle. These are represented in fig. 16. Since any triangular region is projectively transformable into any other, it follows that any triangular region may be represented like Region I in fig. 16. In fig. 18 the four regions are thus represented, together with a portion of the relations among them.

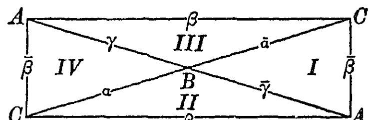


FIG. 18

The representation is more complete if the two segments labeled $\overline{\beta}$ are superposed in such a way that the end labeled A of one coincides with the end labeled A of the other. This is represented in fig. 19 and may be realized in a model by cutting out a rectangular strip of paper, giving it a half twist, and pasting together the two ends.

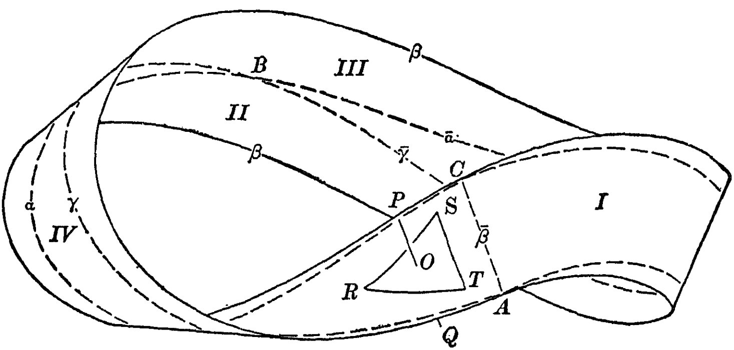


FIG. 19

To complete the model it would be necessary to bring the two edges labeled β in fig. 18 into coincidence. This, however, is not possible in a finite three-dimensional figure without letting the surface cut itself.†

The twisted strip as an example of a one-sided surface is due to Möbius.‡ It has only *one* boundary $A\beta C\beta A$. An imaginary man OP on the surface (fig. 19) could walk, without crossing the boundary, along a path which is the

* It would not be difficult to give a rigorous treatment of the propositions in this section, but it is thought better to postpone this to a later chapter.

image of a straight line in the projective plane, till he arrived at the antipodal position OQ. If a small triangle RST were to be moved with the man without being lifted from the surface or being allowed to touch the man, it would be found, when the man arrived at the position OQ, that the triangle could be superposed upon itself, R coinciding with itself, *but S and T interchanged.* In other words, the boundary of the triangular region containing O would coincide with itself with sense reversed.

It is not essential that the triangular region RST be small, but merely that the figure $ORST$ move continuously so that the triangle RST remains a triangle and the point O is never on one of its sides. The possibility of making this transformation of the figure $ORST$ into $ORTS$ is not affected by joining the two β-edges together, because none of the paths need meet the boundary of the strip. Therefore a corresponding continuous deformation can be made in the projective plane.

If we think of the figure $ORST$ in the projective plane, the four points enter symmetrically. Thus, since S and T can be interchanged by continuously moving the complete quadrangle, any two vertices can be interchanged by such a motion, and hence any permutation of the four vertices can be effected by such a motion. This is intimately associated with the fact that all projectivities in the plane are direct (§ 32), as will be proved in a later chapter, where the notion of continuous deformation of a complete quadrangle in a projective plane is given a precise formulation.

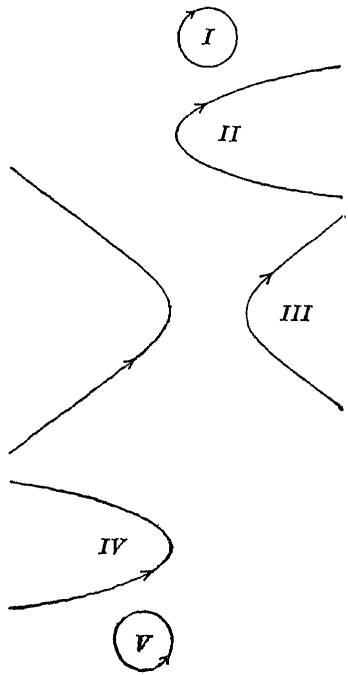


FIG. 20

The triangle RST may be replaced by any small circuit containing O, and it still remains true that O and the circuit may be continuously deformed till O coincides with itself and the circuit coincides with itself reversed. For example, the circuit may be taken as a conic section, and the projective plane imaged as the plane of elementary geometry plus "a line at infinity" (see the introduction to Vol. I, §§ 3, 4, 5, and also § 28 above). The ellipse I (fig. 20) may be deformed into the parabola II, this into the hyperbola III, this into the parabola IV, and this into the ellipse V. The reader can easily verify that the sense indicated by the arrow on I goes continuously to that indicated on V. The figures may be regarded as the projections from a variable center

This deformation of an ellipse and also the corresponding one of the quadrangle $ORST$ depend on internal properties of the surface; i.e. they are independent of the situation of the surface in a three-dimensional space. They are sharply to be distinguished from the property expressed by saying that the man OP comes back to the position OQ, for the latter is a property of the space in which the surface lies.* In fact, the closely related proposition, that if the man OP walk along a straight line in a projective plane till he comes back to the position OQ, the triangle RST comes back to RTS, implies that if a tetrahedron (e.g. $PQRS$) be deformed into coincidence with itself so that two vertices are interchanged, the other two vertices will also be interchanged. And the last statement is a manifestation of the theorem (§ 32) that although the projective plane is one-sided, the projective three-space is two-sided.

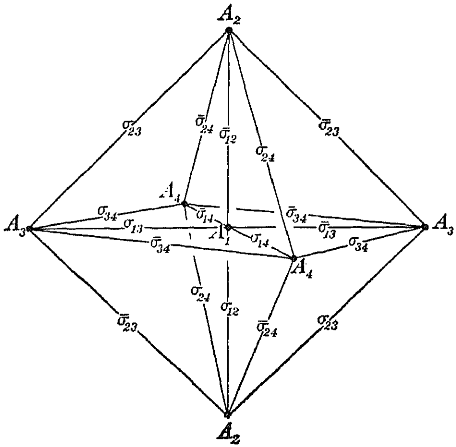


FIG. 21

A sort of model of the projective three-space may be obtained by generalizing the discussion of the plane given above. Any one of the eight regions determined by a tetrahedron is projectively equivalent to any other. Hence we pass from fig. 17 to fig. 21, which represents in full only the relations among the segments, triangular regions, and tetrahedral regions having A_1 as an end, or vertex. Each of the triangles having A_2, A_3, A_4 as vertices is represented by two triangles in fig. 21. Thus, in order to represent the projective space completely we should have to bring each of the triangular regions $A_2A_3A_4$ into coincidence with the one which is symmetrical with it with respect to A_1. In other words, fig. 21 would represent a projective three-space completely if each point on the octahedral surface formed by the triangular regions $A_2A_3A_4$ were brought into coincidence with the opposite point.

EXERCISE

Show that the octahedron in fig. 21 may be distorted into a cube so that the projective three-space is represented by a cube in which each point coincides with its symmetric point with respect to the center of the cube.

* E. Steinitz, Sitzungsberichte der Berliner Mathematischen Gesellschaft, Vol. VII (1908), p. 35.

CHAPTER III

THE AFFINE GROUP IN THE PLANE

34. The geometry corresponding to a given group of transformations. The theorems which we have hitherto considered, whether in general projective geometry or in the particular geometry of reals, state properties of figures which are unchanged when the figures are subjected to collineations. For example, we have had no theorems about individual triangles, because any two triangles are equivalent under the general projective group, and thus are not to be distinguished from one another. On the other hand, there does not, in general, exist a collineation carrying a given pair of coplanar triangles into another given pair of coplanar triangles; and thus we have the theorem of Desargues, and other theorems, stating projective properties of pairs of triangles. We have thus considered only very general properties of figures, and so have dealt hardly at all with the familiar relations, such as perpendicularity, parallelism, congruence of angles and segments, which make up the bulk of elementary Euclidean geometry. These properties are not invariant under the general projective group, but only under certain subgroups. We shall therefore approach their study by a consideration of the properties of these subgroups.

There are, in general, at least two groups of transformations to consider in connection with a given geometrical relation: (1) a group by means of which the relation may be defined, and (2) a group under which the relation is left invariant. These two groups may or may not be the same.*

We have already had one example of a definition of a geometrical relation by means of a group of transformations. In § 19 two collinear triads of points are defined as being in the same sense-class if they are conjugate under the group of direct projectivities on the line. The relation between pairs of triads which is thus defined is invariant under the group of all projectivities (§ 18).

* The group (1) will always be a self-conjugate subgroup of (2), as follows directly from the definition of a self-conjugate subgroup. See § 39, below, where the rôle of

Programm,* *a geometry.* Obviously, all the theorems of the geometry corresponding to a given group continue to be theorems in the geometry corresponding to any subgroup of the given group; and the more restricted the group, the more figures will be distinct relatively to it, and the more theorems will appear in the geometry. The extreme case is the group corresponding to the identity, the geometry of which is too large to be of consequence.

For our purposes we restrict attention to groups of projective collineations,† and in order to get a more exact classification of theorems we narrow the Kleinian definition by assigning to the geometry corresponding to a given group only the theory of those properties which, while invariant under this group, are *not invariant under any other group of projective collineations containing it.* This will render the question definite as to whether a given theorem belongs to a given geometry.

Perhaps the simplest example of a subgroup of the projective group in a plane is the set of all projective collineations which leave a line of the plane invariant. The present chapter is concerned chiefly with the geometry belonging to this group.

The chapter is based entirely on Assumptions A, E, P, H_0. In fact, the theorems of §§ 36, 38, 39, 40, 42, 45, 46, 48 depend only on A, E, H_0. The class of theorems which depend on assumptions with regard to order relations has already been touched on in §§ 28–30.

35. Euclidean plane and the affine group. Let l_∞ be an arbitrary but fixed line of a projective plane π. In accordance with the definition in § 28 we shall refer to l_∞ as the *line at infinity.* The points of l_∞ shall be called *ideal*‡ *points* or *points at infinity,* whereas the remaining points and lines of π shall be called *ordinary* points and lines. The set of all ordinary points is a *Euclidean plane.* In the rest of this chapter the term "point," when unmodified, will refer to an ordinary point.

* Cf. F. Klein, Vergleichende Betrachtungen über neuere geometrische Forschungen, Erlangen 1872; also in Mathematische Annalen, Vol. XLIII (1893), p. 63.

† From some points of view it would have been desirable to include also all projective groups containing correlations.

‡ There is some divergence in the literature with respect to the use of this word and the word "improper." On the latter term see § 85, Vol. I.

DEFINITION. Any projective collineation transforming a Euclidean plane into itself is said to be *affine*; the group of all such collineations is called the *affine group*, and the corresponding geometry the *affine geometry*.

THEOREM 1. *There is one and only one affine collineation transforming three vertices A, B, C of a triangle to three vertices A', B', C' respectively of a triangle.*

Proof. Since l_∞ is transformed into itself, this is a corollary of Theorem 18, § 35, Vol. I.

With respect to any system of nonhomogeneous coördinates of which l_∞ is the singular line, any affine collineation may be written in the form (§ 67, Vol. I)

$$(1) \qquad \begin{aligned} x' &= a_1x + b_1y + c_1, \\ y' &= a_2x + b_2y + c_2, \end{aligned}$$

where

$$\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0.$$

36. Parallel lines. DEFINITION. Two ordinary lines not meeting in an ordinary point are said to be *parallel* to each other, and the pair of lines is said to be *parallel*. A line is also said to be *parallel* to itself.

Hence, in a Euclidean plane we have the following theorem as a consequence of the theorems in Chap. I, Vol. I:

THEOREM 2. *In a Euclidean plane, two points determine one and only one line; two lines meet in a point or are parallel; two lines parallel to a third line are parallel to each other; through a given point there is one and only one line parallel to a given line l.*

DEFINITION. A simple quadrangle $ABCD$ such that the side AB is parallel to CD and BC to DA is called a *parallelogram*.

DEFINITION. The lines AC and BD are called the *diagonals* of the simple quadrangle $ABCD$.

In terms of parallelism, most projective theorems lead to a considerable number of special cases. Moreover, since the affine geometry is not self-dual, theorems which are dual in projective geometry may have essentially different affine special cases. A few affine theorems which are obtainable by direct specialization are given in the following list of exercises, and a larger number in the next section

EXERCISES

1. If the sides of two triangles are parallel by pairs, the lines joining corresponding vertices meet in a point or are parallel.

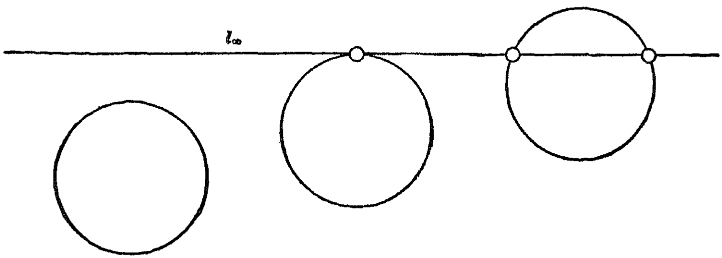
2. If in two projective flat pencils three pairs of corresponding lines are parallel, then each line is parallel to its homologous line.

3. With respect to any system of nonhomogeneous coördinates in which l_∞ is the singular line, the equation of a line parallel to $ax + by + c = 0$ is $ax + by + c' = 0$.

4. A homology (or an elation) whose center and axis are ordinary transforms l_∞ into a line parallel to the axis.

5. If the number of points on a projective line is $p + 1$, the number of points in a Euclidean plane is p^2, the number of triangles in a Euclidean plane is $p^3(p-1)^2(p+1)/6$, and the latter is also the number of projective collineations transforming a Euclidean plane into itself.

37. Ellipse, hyperbola, parabola. Definition. A conic meeting l_∞ in two distinct points is called a *hyperbola*, one meeting it in only one point a *parabola*, and one meeting it in no point an *ellipse*. The



Ellipse *Parabola* *Hyperbola*

Fig. 22

pole of l_∞ is called the *center* of the conic. Any line through the center is called a *diameter*. The tangents to a hyperbola at its points of intersection with l_∞ are called its *asymptotes*. A conic having an ordinary point as center is called a *central conic*.

EXERCISES

1. An ellipse or a hyperbola is a central conic, but a parabola is not.

2. The center of a parabola is its point of contact with l_∞.

3. No two tangents to a parabola are parallel.

4. The asymptotes of a hyperbola meet at its center.

5. Two conjugate diameters (cf. § 44, Vol. I) of a hyperbola are harmonically conjugate with respect to the asymptotes.

6. If a simple hexagon be inscribed in a conic in such a way that two of its pairs of opposite sides are parallel, the third pair of opposite sides is parallel.

7. If a parallelogram be inscribed in a conic, the tangents at a pair of opposite vertices are parallel.

8. If the vertices of a triangle are on a conic and two of the tangents at the vertices are parallel to the respectively opposite sides, the third tangent is parallel to the third side.

9. If a parallelogram be circumscribed to a conic, its diagonals meet in the center and are conjugate diameters.

10. If a parallelogram be inscribed in a conic, any pair of adjacent sides are parallel to conjugate diameters. Its diagonals meet at the center of the conic.

11. Let P and P' be two points which are conjugate with respect to a conic, let p be the diameter parallel to PP', and let Q and Q' be points of intersection with the conic of the diameter conjugate to p. The lines PQ and $P'Q'$ meet on the conic.

12. If a parallelogram $OAPB$ is such that the sides OA and OB are conjugate diameters of a hyperbola and the diagonal OP is an asymptote, then the other diagonal AB is parallel to the other asymptote.

13. If two lines OA and OB are conjugate diameters of a conic which they meet in A and B, then any two parallel lines through A and B respectively meet the conic in two points A' and B' such that OA' and OB' are conjugate diameters.

14. Any two parabolas are conjugate under a collineation transforming l_∞ into itself.*

15. Any two hyperbolas are conjugate under a collineation transforming l_∞ into itself.*

16. Derive the equation of a parabola referred to a nonhomogeneous coördinate system with a tangent and a diameter as axes.

17. Derive the equation of a hyperbola referred to a nonhomogeneous coördinate system with the asymptotes as axes.

18. Derive the equation of an ellipse or a hyperbola referred to a nonhomogeneous coördinate system with a pair of conjugate diameters as axes.

38. The group of translations. DEFINITION. Any elation having l_∞ as an axis is called a *translation.* If l is any ordinary line through the center of a translation, the translation is said to be *parallel* to l.

COROLLARY. *A translation carries every proper line into a parallel line and leaves invariant every line of a certain system of parallel lines.*

THEOREM 3. *There is one and only one translation carrying a point A to a point B.*

Proof. Any translation carrying A to B must be an elation with l_∞ as axis and the point of intersection of the line AB with l_∞ as center. Hence the theorem follows from Theorem 9, Chap. III, Vol. I.

THEOREM 4. *An ordered point pair AB can be carried by a translation to an ordered point pair $A'B'$ such that A' is not on the line AB, if and only if $ABB'A'$ is a parallelogram.*

Proof. Let L_∞ and M_∞ be the points at infinity on the lines AA' and AB respectively. The translation carrying A to A' must carry the line AM_∞ to $A'M_\infty$ and leave the line BL_∞ invariant. Hence the point B, which is the intersection of AM_∞ with BL_∞, is carried to B', which is the intersection of $A'M_\infty$ with BL_∞. Hence the points A' and B' to which A and B respectively are carried by a translation are such that $ABB'A'$ is a parallelogram. Since there is one and only one translation carrying A to A', the same reasoning shows that whenever $ABB'A'$ is a parallelogram there exists a translation carrying A and B to A' and B' respectively.

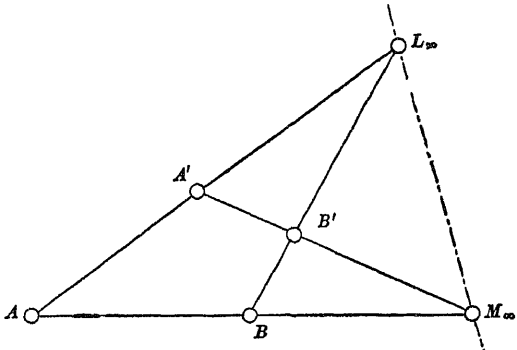


FIG. 23

THEOREM 5. *An ordered point pair AB is carried by a translation to an ordered point pair $A'B'$, where A' is on the line AB, if and only if* $Q(L_\infty AA', L_\infty B'B)$, *$L_\infty$ being the point at infinity of AB.*

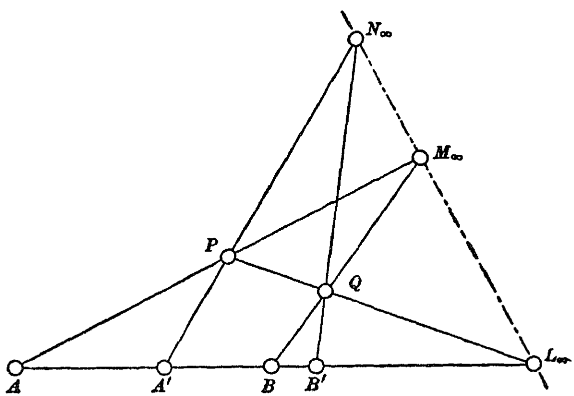


FIG. 24

Proof. Let P be any point not on the line AB, and let M_∞ and N_∞ respectively be the points of intersection of PA and PA' with l_∞. Let Q be the point of intersection of BM_∞ with PL_∞. Then, by the last theorem, the translation carrying

A to B carries P to Q, and hence carries A' to the point of intersection of QN_∞ with AB. Hence N_∞, Q, and B' are collinear, and hence we have $\mathrm{Q}(L_\infty AA', L_\infty B'B)$.

THEOREM 6. *If A, B, C are any three points, the resultant of the translations carrying A to B and B to C is the translation carrying A to C.*

Proof. Let A_∞, B_∞, C_∞ be the points of intersection of the lines BC, CA, AB respectively with l_∞. Suppose first that the three points A_∞, B_∞, C_∞ are all distinct. The translation carrying A to B changes the line AB_∞ into the line BB_∞, and the translation carrying B to C changes the line BB_∞ into CB_∞. Hence the line AB_∞ is invariant under the resultant of these two translations.

Consider now any other line through B_∞, and let it meet AA_∞ in A' and BC in C'; also let B' be the point of intersection of $A'C_\infty$ with BC (fig. 25). We then have that the translation carrying A to B carries A' to B' (Theorem 4), and on account of $\mathrm{Q}(A_\infty BB', A_\infty C'C)$ (Theorem 5) the translation carrying B to C carries B' to C'. Hence the resultant of the two translations carries A' to C' and thus leaves the line $A'B_\infty$ invariant; that is, it leaves all the lines through B_∞ invariant. Since it obviously leaves all points on l_∞ invariant, it is a translation (Cor. 3, Theorem 9, Chap. III, Vol. I).

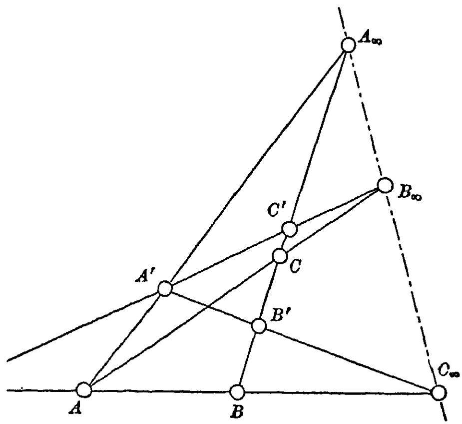


FIG. 25

If two of the three points A_∞, B_∞, C_∞ coincide, they all coincide, and in this case the theorem is obvious.

By definition, the identity is a translation. Hence we have

COROLLARY. *The set of all translations form a group.*

THEOREM 7. *The group of translations is commutative.*

Proof. Given two translations T_1 and T_2 and let A be any point, $\mathrm{T}_1(A) = A'$ and $\mathrm{T}_2(A') = B'$. If $B' = A$, T_2 is the inverse of T_1, and hence T_1 and T_2 are obviously commutative. If $B' \neq A$ and B' is not

on the line AA', let B (fig. 23) be the point of intersection of the line through A parallel to $A'B'$ with the line through B' parallel to AA', then $ABB'A'$ is a parallelogram, and it is obvious that $\mathrm{T}_1(B) = B'$ and $\mathrm{T}_2(A) = B$. Hence $\mathrm{T}_1\mathrm{T}_2(A) = B'$. But, by the definition of A' and B', $\mathrm{T}_2\mathrm{T}_1(A) = B'$. Hence, in this case also, T_1 and T_2 are commutative.

In case B' is on the line AA', let P and Q (fig. 24) be two points such that $A'B'QP$ is a parallelogram, let B be the point of intersection of AA' with the line through Q parallel to AP, and let L_∞, M_∞, N_∞ be the points at infinity of PQ, PA, and PA' respectively. Then, since $\mathrm{T}_2(A') = B'$, it is obvious that $\mathrm{T}_2(P) = Q$, and hence that $\mathrm{T}_2(A) = B$. Moreover, on account of $\mathrm{Q}(L_\infty AB, L_\infty B'A')$, $\mathrm{T}_1(A) = A'$ implies that $\mathrm{T}_1(B) = B'$. Hence $\mathrm{T}_1\mathrm{T}_2(A) = B'$, and thus, in this case also, T_1 and T_2 are commutative.

THEOREM 8. *If OX and OY are two nonparallel lines and* T *is any translation, there is a unique pair of translations* T_1, T_2 *such that* T_1 *is parallel to OX,* T_2 *parallel to OY, and* $\mathrm{T}_1\mathrm{T}_2 = \mathrm{T}$.

Proof. In case T is parallel to OX or OY the theorem is trivial. If T is parallel to neither of them, let $P = \mathrm{T}(O)$ and let X_1 and Y_1 be the points in which the lines through P parallel to OY and OX respectively meet OX and OY respectively. Then OX_1PY_1 is a parallelogram, and if T_1 be the translation carrying O to X_1, and T_2 the translation carrying O to Y_1, it follows, by Theorems 4 and 6, that $\mathrm{T}_1\mathrm{T}_2 = \mathrm{T}$.

On the other hand, if T_1' is any translation parallel to OX, and T_2' any translation parallel to OY, and $\mathrm{T}_1'(O) = X_1'$ and $\mathrm{T}_2'(O) = Y_1'$, the product $\mathrm{T}_1'\mathrm{T}_2'$ carries O to a point P' such that $OX_1'P'Y_1'$ is a parallelogram. But $P' = P$ if and only if $X_1' = X_1$ and $Y_1' = Y_1$. Hence T determines T_1 and T_2 uniquely.

THEOREM. 9. *With respect to a nonhomogeneous coördinate system in which l_∞ is the singular line a translation parallel to the x-axis has the equations*

$$x' = x + a,\quad y' = y. \tag{2}$$

Proof. The point into which $(0, 0)$ is transformed by a given translation parallel to the x-axis may be denoted by $(a, 0)$. By Theorem 5 and § 48, Vol. I, it then follows that any point $(x, 0)$ of the x-axis

the given form (2).

Conversely, any transformation of the type (2) leaves all lines parallel to the x-axis invariant and transforms any other line into a line parallel to itself. Hence it is a translation parallel to the x-axis.

THEOREM 10. *With respect to a nonhomogeneous coördinate system in which l_∞ is the singular line, any translation can be expressed in the form*

$$(3)\qquad \begin{aligned} x' &= x + a, \\ y' &= y + b. \end{aligned}$$

Proof. By Theorem 8 any translation is the product of a translation parallel to the x-axis by one parallel to the y-axis. Hence it is the product of a transformation of the form

$$\begin{aligned} x' &= x + a, \\ y' &= y, \end{aligned}$$

by a transformation of the form

$$\begin{aligned} x' &= x, \\ y' &= y + b. \end{aligned}$$

EXERCISE

Investigate the subgroups of the group of translations.

39. Self-conjugate subgroups. Congruence. DEFINITION. Any subgroup G' of a group G is said to be *self-conjugate* or *invariant** under G if and only if $\Sigma \mathrm{T} \Sigma^{-1}$ is an operation of G' whenever Σ is an operation of G and T of G'.

The geometric significance of this notion is as follows: Suppose that two figures F_1 and F_2 are conjugate under G', and T is a transformation of G' such that $F_2 = \mathrm{T}(F_1)$. If F_1 and F_2 are changed into F_1' and F_2' by any transformation Σ of G, then $\Sigma^{-1}(F_1') = F_1$. Hence† $\mathrm{T}\Sigma^{-1}(F_1') = F_2$,

$$\begin{tikzcd} F_1 \arrow[r, "\Sigma"] \arrow[d, "T"'] & F_1' \arrow[d, "\Sigma T \Sigma^{-1}"] \\ F_2 \arrow[r, "\Sigma"'] & F_2' \end{tikzcd}$$

FIG. 26

* These terms have already been defined in § 75, Vol. I.

† These relations may be illustrated by the accompanying diagram (probably due to S. Lie).

nd $\Sigma T \Sigma^{-1}(F_1') = F_2'$. Therefore, if G' is self-conjugate under G, the gures F_1' and F_2' are conjugate under G'. Hence *the property of eing conjugate under the self-conjugate subgroup G' is a property eft invariant by the group G.* Thus the theory of figures conugate under G' belongs to the geometry corresponding to G, proided that G is not a self-conjugate subgroup of any other group f projective collineations.

THEOREM 11. *The group of translations is self-conjugate under the ffine group.*

Proof. Let T be an arbitrary translation and Σ an arbitrary affine ransformation. We have to show that $\Sigma T \Sigma^{-1}$ is a translation. If ᵖ be any point of l_∞, $\Sigma(P)$ is also on l_∞. Therefore, since T eaves all points of l_∞ invariant, so does $\Sigma T \Sigma^{-1}$. The system of ines through the center of T is a system of parallel lines; Σ transorms this system of parallel lines into a system of parallel lines; nd hence the latter system of parallel lines is invariant under $\Sigma T \Sigma^{-1}$. Hence (cf. Cor. 3, Theorem 9, Chap. III, Vol. I) $\Sigma T \Sigma^{-1}$ is translation.

COROLLARY 1. *The group of translations is self-conjugate under ny subgroup of the affine group which contains it.*

COROLLARY 2. *For any affine collineation Σ, and any translation* T, *here exists a translation* T$'$ *such that* $\Sigma T = T'\Sigma$ *and a translation* T$''$ *uch that* $T\Sigma = \Sigma T''$.

Proof. Let $\Sigma T \Sigma^{-1} = T'$ and $\Sigma^{-1} T \Sigma = T''$. By the theorem, T$'$ and T$''$ are translations. But

$$\Sigma T \Sigma^{-1} = T' \quad \text{and} \quad \Sigma^{-1} T \Sigma = T''$$

mply

$$\Sigma T = T'\Sigma \quad \text{and} \quad T\Sigma = \Sigma T'' \text{ respectively.}$$

DEFINITION. Two figures are said to be *congruent* if they are conugate under the group of translations.

This definition will presently be extended by giving other condi-

40. Congruence of parallel point pairs. The figure consisting c two distinct points A, B may be looked at in two ways with respec to congruence. We consider either the two ordered* point pair AB and BA or the point pair AB without regard to order. In th second case AB and BA mean the same thing and AB is congruen to BA because the identity belongs to the group of translation: On the other hand, the ordered pair AB is not conjugate to th ordered pair BA under the group of translations, because the trans lation carrying A to B does not carry B to A (this is under Assumr tions A, E, H_0).

THEOREM 12. *If $ABDC$ is a parallelogram, the ordered point pai AB is congruent to the ordered point pair CD. If the conditio* Q$(P_\infty AC, P_\infty DB)$ *is satisfied where P_∞ is an ideal point, the ordere point pair AB is congruent to the ordered point pair CD.*

Proof. This is a corollary of Theorems 4 and 5.

COROLLARY 1. *Let A and B be any two distinct points and O th harmonic conjugate of the point at infinity of the line AB with respec to A and B. Then the pair AO is congruent to the pair OB.*

DEFINITION. The point O in the last corollary is called the *mic point* of the pair AB. In case $B = A$, A is called the *mid-point* c the pair AB.

COROLLARY 2. *The line joining the mid-points of the pairs of vertice AB and AC of a triangle ABC is parallel to the line BC.*

Proof. Let B_∞ and C_∞ be the points at infinity of the lines AB an AC respectively, and let B_1 and C_1 be the mid-points of the pairs A and AC respectively. Then, by the definition of "mid-point,"

$$AB_1BB_\infty \doublebarwedge AC_1CC_\infty.$$

Hence the lines B_1C_1, BC, and $B_\infty C_\infty$ concur, which means that B_1C and BC are parallel.

DEFINITION. The line joining a vertex, say A, of a triangle ABC t the mid-point of BC is called a *median* of the triangle.

THEOREM 13. *The three medians of a triangle meet in a point.*

* Cf. footnote on page 40.

Proof. Let the triangle be ABC; let A_∞, B_∞, C_∞ be the points at infinity of the sides BC, CA, AB respectively; and let A_1, B_1, C_1 be the points of intersection of the pairs of lines BB_∞ and CC_∞, CC_∞ and AA_∞,

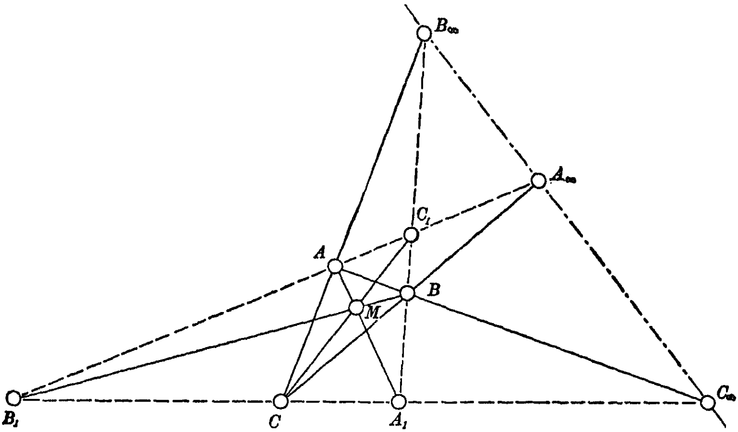


FIG. 27

AA_∞ and BB_∞ respectively (fig. 27). Then, by well-known theorems on harmonic sets (§ 31, Vol. I), the medians of the triangle ABC are AA_1, BB_1, and CC_1, and these three lines concur.

EXERCISES

1. The diagonals of a parallelogram bisect one another; that is, if $ABCD$ is a parallelogram, the mid-points of the pairs AC and BD coincide.

2. Let a and b be two parallel lines. The mid-points of all the pairs AB where A is on a and B on b are on a line parallel to a and b.

3. If the sides AB, BC, CA of a triangle ABC are respectively parallel to the sides $A'B'$, $B'C'$, $C'A'$ of a triangle $A'B'C'$, and the ordered point pair AB is congruent to the ordered point pair $A'B'$, then the two triangles are congruent.

4. The mid-points of the pairs of opposite vertices of a complete quadrilateral are collinear. Let us call this line the *diameter* of the quadrilateral.

5. A line through a diagonal point O of a complete quadrangle, parallel to the opposite side of the diagonal triangle, is met by either pair of opposite sides of the quadrangle which do not pass through O in a pair of points having O as mid-point.

41. Metric properties of conics. The following list of exercises contains a number of theorems on conics which involve the congru-

EXERCISES

1. The mid-points of a system of pairs of points of a conic AA', BB', CC', etc. are collinear if the lines AA', BB', CC' are parallel. The line containing the mid-points is a diameter conjugate to the diameter parallel to AA'.

2. Let A and B be two points of a parabola. If the line joining the mid-point C of the pair AB to the pole P of the line AB meets the conic in O, then O is the mid-point of the pair CP.

3. If a line meets a hyperbola in a pair of points H_1H_2, and its asymptotes in a pair A_1A_2, the two pairs have the same mid-point. The pair H_1A_1 is congruent to the pair H_2A_2.

4. The point of contact of a tangent to a hyperbola is the mid-point of the pair in which the tangent meets the asymptotes.

5. Let A_1 and A_2 be each a fixed and X a variable point of a hyperbola, and let X_1 and X_2 be the points in which the lines XA_1 and XA_2 meet one of the asymptotes. The point pairs X_1X_2 determined by different values of X are all congruent.

6. The centers of all conics inscribed in* a simple quadrilateral $ABCD$ are on the line joining the mid-points of the point pairs CA and BD.

7. The centers of all conics which pass through the vertices of a complete quadrangle $ABCD$ are on a conic C^2, which contains the six mid-points of the pairs of vertices of the quadrangle, the three vertices of its diagonal triangle, and the double points (if existent) of the involution in which l_∞ is met by the pencil of conics through A, B, C, D. From the projective point of view, according to which l_∞ is any line whatever, C^2 is called the *nine-point* (or the *eleven-point*) *conic* of the complete quadrangle $ABCD$ and the line l_∞. Derive the analogous theorems for the pencils of conics of Types II–V (cf. § 47, Vol. I).

8. The five diameters† of the complete quadrilaterals formed by leaving out one line at a time from a five-line meet in a point A, which is the center of the conic tangent to the five lines.

9. The six points A determined, according to the last exercise, by the six complete five-lines formed by leaving out one line at a time from a six-line are on a conic C^2.

10. The seven conics C^2 determined, according to the last exercise, by the seven complete six-lines formed by leaving out one line at a time from a seven-line, all pass through three points.

42. Vectors. Any ordered pair of points determines a set of pairs all of which are equivalent to it under the group of translations. In order to study the relations between such sets of pairs we introduce the notion of a vector. The term "vector" appears in the literature

* A conic is said to be inscribed in a given figure if the figure is circumscribed to the conic (cf. § 43, Vol. I).

the term "ordered pair of points" is to be understood to include the case of a single point counted twice.

DEFINITION. *A planar field of vectors* (or *vector field*) is any set of objects, the individuals of which are called *vectors*, such that (1) there is one vector for each ordered pair of points in a Euclidean plane, and (2) there is only one vector for any two ordered pairs AB and $A'B'$ which are equivalent under the group of translations. A vector corresponding to a coincident pair of points is called a *null* vector or a *zero* vector, and denoted by the symbol 0.

For example, a properly chosen set of matrices would be a vector field according to this definition. So would also the set of all translations including the identity; also a set of classes of ordered point pairs such that two point pairs are in the same class if and only if equivalent under the group of translations. However a vector field be defined, it will be found that, in most applications, only those properties which follow from the definition as stated above are actually used.

A precisely similar state of affairs exists in the definition of a number system. The objects in the particular number system determined for a given space by the methods of Chap. VI, Vol. I, are points, but a number system in general is any set of objects in a proper one-to-one correspondence with this set of points.

In the following discussion has been selected, and all stat the vector corresponding tu the point pair AB is a definite object, and we shall denote it as "the vector AB," or, in symbols, Vect (AB).

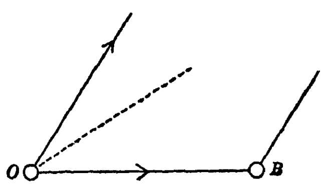


FIG. 28

Since any point of a Euclidean plane can be carried by a translation to any other point, the set of all vectors is the same as the set of vectors OA, where O is a fixed and A a variable point. Consequently, the following definition gives a meaning to the operation of "adding" any two vectors.

DEFINITION. If O, A, C are points of a Euclidean plane, the vector OC is called the *sum* of the vectors OA and AC. In symbols this is

indicated by Vect $(OC) =$ Vect $(OA) +$ Vect (AC). The operation of obtaining the sum of two vectors is called *addition* of vectors.

An obvious corollary of this definition is that

$$\text{Vect}\,(AB) + \text{Vect}\,(BA) = 0.$$

Hence we define:

DEFINITION. The vector Vect (BA) is called the *negative* of the vector Vect (AB), and denoted by $-$ Vect (AB).

THEOREM 14. *The operation of addition of vectors is associative; that is, if a, b, c are vectors, $(a + b) + c = a + (b + c)$.*

Proof. Let the three vectors be OA, AB, BC respectively; then, by definition, both (Vect $(OA)+$Vect $(AB))+$Vect (BC) and Vect $(OA)+$ (Vect $(AB)+$Vect (BC)) are the same as Vect (OC).

DEFINITION. Two vectors are said to be *collinear* if and only if they can be expressed as Vect (OA) and Vect (OB) respectively, where O, A, B are collinear points.

THEOREM 15. *The sum of two noncollinear vectors OA and OB is the vector OC, where C is such that $OACB$ is a parallelogram.*

Proof. By Theorem 4, the vector OB is the same as the vector AC. Hence, by definition, the sum of OA and OB is OC.

THEOREM 16. *The sum of two collinear vectors OA and OB is a vector OC such that* $Q(P_\infty AO, P_\infty BC)$, *where P_∞ is the point at infinity of the line AB.*

Proof. Let L and M be two points such that $OBML$ is a parallelogram. Hence Vect $(OB) =$ Vect (LM). Then, by definition, C must be such that Vect $(LM) =$ Vect (AC), that is, such that $ACML$ is a parallelogram. Let L_∞ be the ideal point of intersection of the lines OL and BM, and let M_∞ be the ideal point of intersection of the lines AL and MC. The complete quadrangle

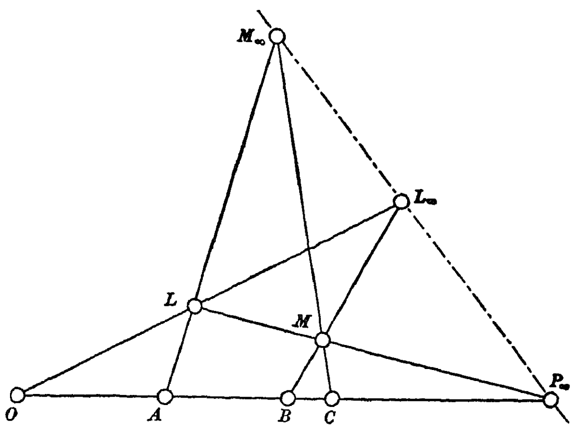


FIG. 29

such that $Vect(OA) + Vect(OB) = Vect(OC)$, *then, with respect to any scale* (cf. § 48, Vol. I) *in which* P_0 *is* O *and* P_∞ *the point at infinity of the line* OA,

$$A + B = C.$$

Proof. Cf. Cor. 1, Theorem 1, Chap. VI, Vol. I.

THEOREM 17. *The operation of adding vectors is commutative; that is, if* a *and* b *are vectors,* $a + b = b + a$.

Proof. Let the vectors a and b be $\text{Vect}(OA)$ and $\text{Vect}(OB)$ respectively. If O, A, B are noncollinear, the result follows from Theorem 15, and if they are collinear, from Theorem 16.

43. Ratios of collinear vectors. By analogy with the case of addition we should be led to base a definition of multiplication of collinear vectors upon the multiplication of points in § 49, Vol. I. There are, however, a great many ways of defining the product of two vectors, which would not reduce to this sort of multiplication in the case of collinear vectors. Hence, in order to avoid possible confusion we shall not introduce a definition of the multiplication of vectors at present, but only of what we shall call the ratio of two collinear vectors.

DEFINITION. The *ratio* of two collinear vectors OA and OB is the number which corresponds to A in the scale in which P_0 is O, P_1 is B, and P_∞ is the point at infinity of the line OA. It is denoted by

$$\frac{\text{Vect}(OA)}{\text{Vect}(OB)} \text{ or by } \frac{OA}{OB}.$$

It is to be emphasized that the ratio of two collinear vectors as here defined is a number. By comparison with the definition in § 56, Vol. I, we have at once

THEOREM 18. *If* A, B, C, D_∞ *are collinear points,* D_∞ *being ideal,*

$$\text{R}(D_\infty A, BC) = \frac{AC}{AB}.$$

Theorem 13, Chap. VI, Vol. I now gives

THEOREM 19. *If* A_1, A_2, A_3, A_4 *are any four collinear ordinary points,*

$$\text{R}(A_1A_2, A_3A_4) = \frac{A_1A_3}{A_1A_4} : \frac{A_2A_3}{A_2A_4}.$$

THEOREM 20. *If two triangles* ABC *and* $A'B'C'$ *are such that the sides* AB, BC, CA *are parallel to* $A'B'$, $B'C'$, $C'A'$ *respectively,*

$$\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CA}{C'A'}.$$

Proof. Suppose that the translation which carries A' to A carries B' to B_1 and C' to C_1. Then B_1 is on the line AB and C_1 on the line AC, and the line B_1C_1 is parallel to BC. Thus, if B_∞ be the point at infinity of the line AB, and C_∞ the point at infinity of the line AC,

$$B_\infty ABB_1 \overline{\wedge} C_\infty ACC_1.$$

Hence, by Theorem 18, $$\frac{AB}{AB_1} = \frac{AC}{AC_1} = \frac{CA}{C_1A},$$

which is, by definition, the same as

$$\frac{AB}{A'B'} = \frac{CA}{C'A'}.$$

In like manner, it follows that

$$\frac{AB}{A'B'} = \frac{BC}{B'C'}.$$

Since we have not defined the product of two vectors, it is necessary to resort to a device in order to compute conveniently with them. This we do as follows:

DEFINITION. With respect to an arbitrary vector OA, which is called a unit vector, the ratio $$\frac{OB}{OA},$$

where OB is any vector collinear with OA, is called the *magnitude* of OB.

Observe that the magnitude of OB is the negative of the magnitude of BO. Since the magnitude of a vector is a number, there is no difficulty about algebraic computations with magnitudes. In the rest of this section we shall use the symbol AB to denote the magnitude of the vector AB. No confusion is introduced by this double use of the symbol, because the ratio of two vectors is precisely the same as the quotient of their magnitudes.

DEFINITION. If Γ is any collineation not leaving l_∞ invariant, the lines $\Gamma(l_\infty)$ and $\Gamma^{-1}(l_\infty)$ are called the *vanishing lines* of Γ. If Π is any projectivity transforming a line l to a line l' (which may coincide with l), the ordinary points of l and l' which are homologous with points at infinity are (if existent) called the *vanishing points* of Π. If Π is an involution transforming l into itself but not leaving the point at infinity invariant, the vanishing point is called the *center* of the involution.

THEOREM 21. DEFINITION. *If O and O' are the vanishing points,*

parallel line l', and X is a variable point of l, and X' the point of l' to which X is transformed, the product $OX \cdot O'X'$ is a constant, called the* power *of the transformation.*

Proof. Let P_∞ be the point at infinity of l and l'; and let X_1 and X_2 be two values of X, and X_1' and X_2' the points to which they are transformed by the given projectivity. Then, by the fundamental property of a cross ratio,

$$\text{R}(P_\infty O, X_1X_2) = \text{R}(O'P_\infty, X_1'X_2') = \text{R}(P_\infty O', X_2'X_1'),$$

and hence, by Theorem 18, $\dfrac{OX_2}{OX_1} = \dfrac{O'X_1'}{O'X_2'}$.

Hence, by the definition of magnitude of vectors,

$$OX_2 \cdot O'X_2' = OX_1 \cdot O'X_1'.$$

COROLLARY 1. *The power of an involution having a center O and a conjugate pair AA_1 is $OA \cdot OA_1$.*

COROLLARY 2. *Let Π be a homology whose center is an ordinary point F and whose axis is an ordinary line, and let D be any point of the vanishing line $\Pi^{-1}(l_\infty)$. If P is a variable point, $P' = \Pi(P)$, and D' is the point in which the line through P' parallel to FD meets the vanishing line $\Pi(l_\infty)$, then* $\dfrac{FP}{FP'} = \dfrac{DF}{P'D'}$.

Proof. Let Q and Q' be the points in which the line FP meets the vanishing lines $\Pi^{-1}(l_\infty)$ and $\Pi(l_\infty)$ respectively. By the theorem,

$$PQ \cdot P'Q' = FQ \cdot FQ';$$

from which we derive successively

$$\frac{PF + FQ}{FQ} = \frac{FP' + P'Q'}{P'Q'},$$

$$\frac{PF}{FQ} = \frac{FP'}{P'Q'},$$

$$\frac{FP}{FP'} = \frac{QF}{P'Q'}.$$

Since Π is a homology, the two vanishing lines are parallel. Hence

$$\frac{QF}{P'Q'} = \frac{DF}{P'D'}.$$

Hence $$\frac{FP}{FP'} = \frac{DF}{P'D'}.$$

EXERCISES

1. If a projectivity $ABCD \overline{\wedge} A'B'C'D'$ is such that the point at infinity of the line AB corresponds to the point at infinity of the line $A'B'$,

$$\frac{AB}{CD} = \frac{A'B'}{C'D'}.$$

2. If three parallel lines a, b, c are met by one line in the points A', B', C' respectively and by another line in $A''B''C''$ respectively, then

$$\frac{A'B'}{A'C'} = \frac{A''B''}{A''C''}.$$

3. If $ABCD$ are any four collinear points,

$$AB \cdot CD + AC \cdot DB + AD \cdot BC = 0.$$

4. Six points form a quadrangular set $Q\,(A_2B_2C_2,\ A_1B_1C_1)$ if and only if

$$\mathrm{R}(A_1A_2,\ B_1C_1) \cdot \mathrm{R}(B_1B_2,\ C_1A_1) \cdot \mathrm{R}(C_1C_2,\ A_1B_1) = -1.$$

5. The condition for a quadrangular set may also be written

$$\frac{A_1B_2}{A_2B_1} \cdot \frac{B_1C_2}{B_2C_1} \cdot \frac{C_1A_2}{C_2A_1} = -1.$$

6. If three tangents to a parabola meet two other tangents in P_1, P_2, P_3 and Q_1, Q_2, Q_3 respectively, then

$$\frac{P_1P_2}{P_1P_3} = \frac{Q_1Q_2}{Q_1Q_3}.$$

Conversely, if five lines are such that the points in which two of them meet the other three satisfy this condition, the conic to which the five lines are tangent is a parabola.

7. Let O be the center of a hyperbola, and A_1 and A_2 the points in which the asymptotes are met by an arbitrary tangent; if another tangent meets the asymptotes OA_1, OA_2 in B_1 and B_2 respectively,

$$\frac{OA_1}{OB_1} = \frac{OB_2}{OA_2}.$$

8. If a fixed tangent p to a conic at a point P meets two variable conjugate diameters in Q and Q', then $PQ \cdot PQ'$ is a constant. Let O be the center of the conic. If the diameter parallel to p meets the conic in S, then

$$PQ \cdot PQ' = -(OS)^2.$$

9. Let O_1 and O_2 be the points of contact of two fixed parallel tangents to a conic. If a variable tangent meets the two fixed tangents in X_1 and X_2 respectively, $O_1X_1 \cdot O_2X_2$ is constant. If O is the center of the conic and B is a point of intersection of the diameter through O parallel to the fixed tangents,

44. Theorems of Menelaus, Ceva, and Carnot.

THEOREM 22 (MENELAUS). *Three points A', B', C' of the sides BC, CA, AB, respectively, of a triangle are collinear if and only if*

$$\frac{A'B}{A'C}\cdot\frac{B'C}{B'A}\cdot\frac{C'A}{C'B}=1.$$

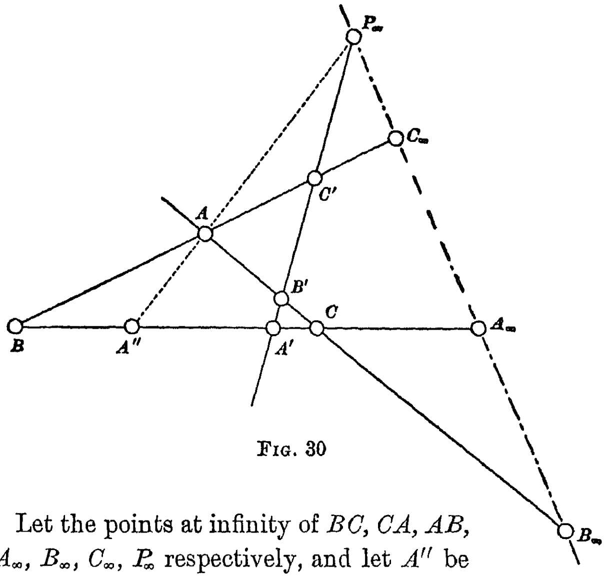


FIG. 30

Proof. Let the points at infinity of BC, CA, AB, $A'B'$ be A_∞, B_∞, C_∞, P_∞ respectively, and let A'' be the intersection of AP_∞ with BC. Then, supposing A', B', C' collinear,

$$(B_\infty B'AC)\overset{P_\infty}{\underset{\wedge}{=}}(A_\infty A'A''C) \text{ and } (C_\infty C'BA)\overset{P_\infty}{\underset{\wedge}{=}}(A_\infty A'BA'').$$

Hence $$\frac{B'C}{B'A}=\mathrm{R}\,(B_\infty B',\ AC)=\mathrm{R}\,(A_\infty A',\ A''C)=\frac{A'C}{A'A''},$$

and $$\frac{C'A}{C'B}=\mathrm{R}\,(C_\infty C',\ BA)=\mathrm{R}\,(A_\infty A',\ BA'')=\frac{A'A''}{A'B}.$$

Hence $$\frac{A'B}{A'C}\cdot\frac{B'C}{B'A}\cdot\frac{C'A}{C'B}=\frac{A'B}{A'C}\cdot\frac{A'C}{A'A''}\cdot\frac{A'A''}{A'B}=1.$$

The converse argument is now obvious.

THEOREM 23 (CEVA). *The necessary and sufficient condition for the concurrence of the lines joining the vertices A, B, C of a triangle to*

Suppose first that C'' is an ordinary point. Then, by the theorem of Menelaus,

$$\frac{A'B}{A'C}\cdot\frac{B'C}{B'A}\cdot\frac{C''A}{C''B}=1. \tag{5}$$

The point C'' is harmonically conjugate to C' with respect to A and B if and only if the lines AA', BB', CC' meet in a point. Thus,

$$\frac{C'A}{C'B}\div\frac{C''A}{C''B}=-1 \tag{6}$$

is a necessary and sufficient condition that AA', BB', CC' concur. But on multiplying (5) by (6) we obtain (4).

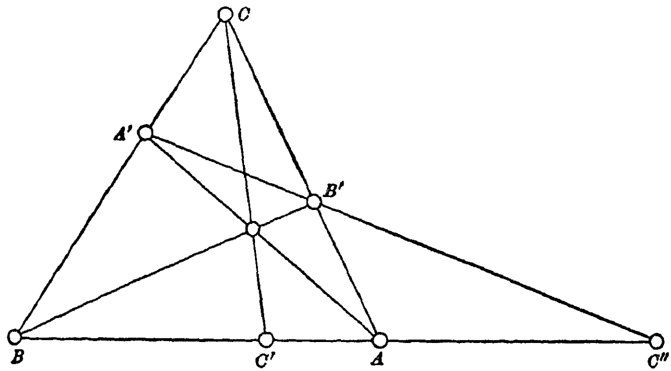


FIG. 31

In case C'' is an ideal point, the line $A'B'$ is parallel to AB and, by Theorem 20,

$$\frac{A'B}{A'C}\cdot\frac{B'C}{B'A}=1. \tag{7}$$

The condition that C'' be harmonically conjugate to C' with regard to A and B now takes the form

$$\frac{C'A}{C'B}=-1.$$

On multiplying this into (7) we again obtain (4).

THEOREM 24 (CARNOT). *Three pairs of points, A_1A_2, B_1B_2, C_1C_2, respectively, on the sides BC, CA, AB, respectively, of a triangle are on the same conic if and only if*

$$\frac{A_1B}{A_1C}\cdot\frac{B_1C}{B_1A}\cdot\frac{C_1A}{C_1B}\cdot\frac{A_2B}{A_2C}\cdot\frac{B_2C}{B_2A}\cdot\frac{C_2A}{C_2B}=1. \tag{8}$$

A_1, B_1, C_1 and A_2, B_2, C_2 respectively. The formula (8) in this case follows directly from Theorem 22 when we multiply together the conditions that A_1, B_1, C_1 and A_2, B_2, C_2 be respectively collinear.

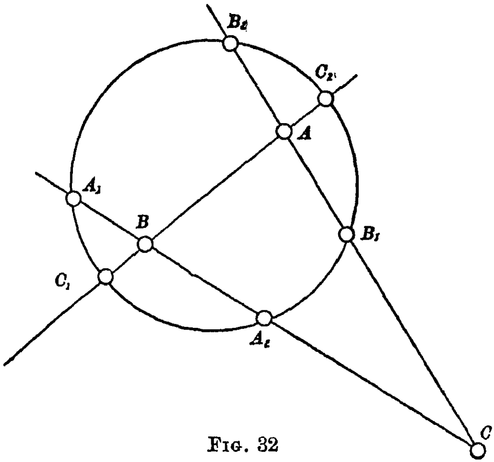


FIG. 32

Now consider any proper conic through A_1, A_2, B_1, B_2 meeting the line AB in $C_1'C_2'$. By the theorem of Desargues (Theorem 19, Chap. V, Vol. I) the pairs AB, C_1C_2, and $C_1'C_2'$ are in involution. Hence

$$C_1C_2'AB \,\overline{\wedge}\, C_2C_1'BA \,\overline{\wedge}\, C_1'C_2AB,$$

and hence
$$\frac{C_1A}{C_1B} : \frac{C_2'A}{C_2'B} = \frac{C_1'A}{C_1'B} : \frac{C_2A}{C_2B},$$

or
$$\frac{C_1A}{C_1B} \cdot \frac{C_2A}{C_2B} = \frac{C_1'A}{C_1'B} \cdot \frac{C_2'A}{C_2'B}.$$

Hence the formula (8) is equivalent to the formula obtained from it by substituting C_1', C_2' for C_1, C_2 respectively. Hence the formula holds for any conic. The converse argument is now obvious.

The last three theorems are the most important special cases of the "theory of transversals." A few further theorems of this class and some other propositions which can readily be derived from them are stated in the exercises below. Further theorems and references will be found in the Encyclopädie der Math. Wiss. III AB 5, § 2, and III C 1, § 23.

EXERCISES

1. The six lines joining the vertices A, B, C of a triangle to pairs of points A_1A_2, B_1B_2, C_1C_2 on the respectively opposite sides are tangents to a conic if and only if the relation (8) is satisfied.

2. If the sides BC, CA, AB of a triangle are tangent to a conic in A_1, B_1, C_1 respectively,

$$\frac{CA_1}{BA_1} \cdot \frac{AB_1}{CB_1} \cdot \frac{BC_1}{AC_1} = -1.$$

3. If a line BC meets a conic in A_1 and A_2, and two parallel lines through B and C, respectively, meet it in the pairs C_1, C_2 and B_1, B_2 respectively,

$$\frac{A_1B}{A_1C}\cdot\frac{A_2B}{A_2C}\cdot\frac{B_1C}{C_1B}\cdot\frac{B_2C}{C_2B}=1.$$

4. Let two lines a and b through a point O meet a conic in the pairs A_1, A_2 and B_1, B_2 respectively. If O, a, b are variable in such a way that a and b remain respectively parallel to two fixed lines,

$$\frac{OA_1\cdot OA_2}{OB_1\cdot OB_2}$$

is a constant.

5. If the sides of a triangle meet a conic in three pairs of points, the three pairs of lines joining the pairs of points to the opposite vertices of the triangle are tangents to a second conic. State the dual and converse of this theorem.

6. If two points are joined to the vertices of a triangle by six lines, these lines meet the sides in six points (other than the vertices) which are on a conic. Dualize.

7. If a line meets the sides $A_0A_1, A_1A_2, \cdots, A_nA_0$, respectively, of a simple polygon $A_0A_1A_2\cdots A_n$ in points $B_0, B_1, \cdots, B_n$ respectively,

$$\frac{A_0B_0}{A_1B_0}\cdot\frac{A_1B_1}{A_2B_1}\cdots\frac{A_nB_n}{A_0B_n}=1.$$

8. If a conic meets the lines $A_0A_1, A_1A_2, \cdots, A_nA_0$, respectively, in the pairs of points $B_0C_0, B_1C_1, \cdots, B_nC_n$ respectively,

$$\frac{A_0B_0}{A_1B_0}\cdot\frac{A_0C_0}{A_1C_1}\cdot\frac{A_1B_1}{A_2B_1}\cdot\frac{A_1C_1}{A_2C_1}\cdots\frac{A_nB_n}{A_0B_n}\cdot\frac{A_nC_n}{A_0C_n}=1.$$

9. If a conic is tangent to the lines $A_0A_1, A_1A_2, \cdots, A_nA_0$, respectively, in the points $B_0, B_1, \cdots, B_n$ respectively,

$$\frac{A_0B_0}{A_1B_0}\cdot\frac{A_1B_1}{A_2B_1}\cdots\frac{A_nB_n}{A_0B_n}=(-1)^{n-1}.$$

45. Point reflections. DEFINITION. A homology of period two whose axis is l_∞ is called a *point reflection.*

From this definition there follows at once:

THEOREM 25. *A point reflection is fully determined by its center. The center is the mid-point of every pair of homologous points. Every two homologous lines are parallel.*

THEOREM 26. *The product of two point reflections whose centers are distinct is a translation parallel to the line joining their centers.*

Proof. The product obviously leaves fixed all points of l_∞ and also

and let P be any point not on the line C_1C_2. Also let P' be the transform of P by the point reflection with C_1 as center, and let Q be the transform of P' by the point reflection with C_2 as center. Since C_1 is the mid-point of the pair PP', and C_2 of the pair $P'Q$, the line PQ is parallel to C_1C_2 (Theorem 12, Cor. 2). Thus the product of the two point reflections leaves invariant all lines parallel to C_1C_2, and hence is a translation.

COROLLARY. *The product of any even number of point reflections is a translation.*

THEOREM 27. *Any translation is the product of two point reflections one of which is arbitrary.*

Proof. Let T be any translation, C_1 the center of any point reflection, $C_3 = \mathrm{T}(C_1)$, and C_2 the mid-point of the pair C_1C_3. The product of the reflections in C_1 and C_2 is a translation, by Theorem 26, and since it carries C_1 to C_3, it is the translation T, by Theorem 3.

COROLLARY 1. *The product of any odd number of point reflections is a point reflection.*

Proof. Let the given point reflections be $\mathrm{P}_1, \mathrm{P}_2, \cdots, \mathrm{P}_{2n+1}$. By Theorem 26 the product $\mathrm{P}_1\mathrm{P}_2 \cdots \mathrm{P}_{2n}$ reduces to a translation, which, by Theorem 27, is the product of two point reflections one of which is P_{2n+1}. Hence there exists a point reflection P such that

$$\mathrm{P}_1\mathrm{P}_2 \cdots \mathrm{P}_{2n+1} = \mathrm{P}\mathrm{P}_{2n+1}\mathrm{P}_{2n+1} = \mathrm{P}.$$

COROLLARY 2. *The product of a translation and a point reflection is a point reflection.*

COROLLARY 3. *The set of all point reflections and translations form a group.*

THEOREM 28. *The group of point reflections and translations is a self-conjugate subgroup of the affine group.*

Proof. It has been proved, in Theorem 11, that if T is a translation and Σ an affine collineation, $\Sigma\mathrm{T}\Sigma^{-1}$ is a translation. Precisely similar reasoning shows that if T is a point reflection, $\Sigma\mathrm{T}\Sigma^{-1}$ is a point reflection.

COROLLARY. *The group G of point reflections and translations is*

THEOREM 29. *With respect to any system of nonhomogeneous coördinates in which l_∞ is the singular line, the equations of a point reflection have the form*

$$(9) \qquad \begin{aligned} x' &= -x + a, \\ y' &= -y + b. \end{aligned}$$

Proof. The point reflection whose center is the origin is of the form

$$\begin{aligned} x' &= -x, \\ y' &= -y, \end{aligned}$$

because this transformation evidently leaves (0, 0) and l_∞ pointwise invariant and is of period two. Since any other point reflection is the resultant of this one and a translation, it must be of the form (9).

EXERCISES

1. An ellipse or a hyperbola is transformed into itself by a point reflection whose center is the center of the conic.

2. Let $[C^2]$ be a system of conics conjugate under the group of translations to a single conic. Under what circumstances is $[C^2]$ invariant under the group of translations and point reflections?

3. Investigate the subgroups of the group of translations and point reflections.

4. Any odd number of point reflections $P_1, P_2, \cdots, P_n$ satisfy the condition,

$$(P_1 P_2 \cdots P_n)^2 = 1.$$

5. Let T be the point reflection whose center is the pole of l_∞ with respect to the n-point whose vertices are the centers of n point reflections $P_1, P_2, \cdots, P_n$. Then*

$$P_1 T P_2 T P_3 T \cdots P_n T = 1.$$

46. Extension of the definition of congruence. DEFINITION. Two figures are said to be *congruent* if they are conjugate under the group of translations and point reflections.

This definition is obviously in agreement with that given in § 39. It will be completed in § 57, Chap. IV. The main significance of the present extension of the definition is that it removes any necessity of distinguishing between ordered and nonordered point pairs in statements about congruence.

* Cf. pp. 46, 84, Vol. I. The center of T is the "center of gravity" of the centers of $P_1, \cdots, P_n$. Cf. H. Wiener, Berichte der Gesellschaft der Wissenschaften zu Leipzig, Vol. XLV (1893), p. 568.

THEOREM 30. *Any ordered point pair AB is congruent to the rdered point pair BA.*

Proof. Let O be the mid-point of the ordered point pair AB. The oint reflection with O as center interchanges A and B.

COROLLARY. *If a point reflection transforms an ordered point pair B to $A'B'$,*

$$\text{Vect}\,(AB) = -\,\text{Vect}\,(A'B').$$

Proof. By Theorem 26 the given point reflection is the product of ne point reflection in the mid-point of AB and a translation. The oint reflection in the mid-point of AB interchanges A and B, and the anslation leaves all vectors unchanged.

47. The homothetic group. DEFINITION. A homology whose axis l_∞ is called a *dilation.* Dilations and translations are both called *omothetic transformations.* Two figures conjugate under a homoetic transformation are said to be *homothetic.*

Homothetic figures are also called, in conformity with definitions troduced later, "similar and similarly placed."

The point reflections are evidently special cases of dilations. Since ne product of two perspective collineations (§ 28, Vol. I) having a ommon axis is a perspective collineation, the set of all homothetic ansformations form a group; and by an argument like that used or Theorem 11 this group is self-conjugate under the affine group. ence we have

THEOREM 31. *The set of all homothetic transformations form a roup which is a self-conjugate subgroup of the affine group.*

Further theorems on the homothetic group are stated in the xercises below.

EXERCISES

1. The ratios of parallel vectors are left invariant by the homothetic group.

2. If two point pairs AB and CD are transformed by a dilation into $A'B'$ nd $C'D'$ respectively,

$$\frac{AB}{A'B'} = \frac{CD}{C'D'}.$$

3. If two triangles are homothetic, the lines joining corresponding vertices eet in a point or are parallel.

4. The equations of the homothetic group with respect to any nonhomoeneous coördinate system of which l_∞ is the singular line are

$$x' = ax + b, \qquad (a \neq 0)$$
$$y' = ay + d.$$

48. Equivalence of ordered point triads. Although the theory of congruence as based on the group of translations and point reflections does not yield metric relations between pairs of points unless they are on parallel lines, yet when applied to point triads it yields a complete theory of the equivalence (in area) of triangles.*

In this section we shall give the definitions and the more important sufficient conditions for equivalence, using methods somewhat analogous to those in the first book of Euclid's Elements. Instead of triangles, however, we shall work with ordered triads of points. This permits the introduction of algebraic signs of areas, though, as we do not need to refer to the interior and exterior of a triangle, we shall not actually employ the word "area." The triads of points which are referred to are all triads of *noncollinear points.*

Our definitions have their origin in the intuitional notions: that any triangle ABC is equivalent in area to the triangle BCA, that two triangles are equivalent in area if one can be transformed into the other by a translation or point reflection, and that two triangles which can be obtained by adding equivalent triangles are equivalent.

DEFINITION. If ABC and ACD are two ordered point triads, and B, C, and D are collinear, and $B \neq D$ (fig. 33), the point triad ABD is called the *sum* of ABC and ACD and is denoted by $ABC + ACD$ or by $ACD + ABC$.

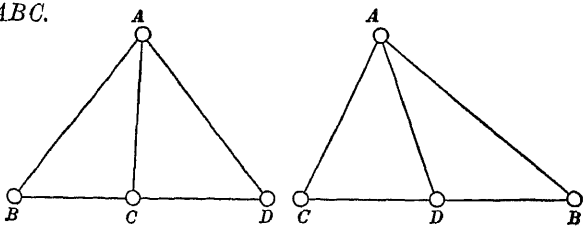


FIG. 33

DEFINITION. An ordered point triad t is said to be *equivalent* to an ordered point triad t' (in symbols, $t \backsimeq t'$) (1) if t can be carried to t' by a point reflection, or (2) if t and t' can be denoted by ABC and

* The idea of building up the theory of areas without the aid of a full theory of congruence is due to E. B. Wilson, Annals of Mathematics, Vol. V (2d series) (1903), p. 29. His method is quite different from ours, being based on the observation (cf. § 52, below) that an equiaffine collineation is expressible as a product of simple shears. Still another treatment of areas based on the group of translations and employing continuity considerations is outlined by Wilson and Lewis, "The Space-time Manifold of Relativity," Proceedings of the American Academy of Arts and

BCA respectively, or (3) if there exists an ordered point triad $\bar{t}$ such that $t \backsimeq \bar{t}$ and $\bar{t} \backsimeq t'$, or (4) if there exist ordered point triads t_1, t_2, t_1', t_2' such that $t_1 \backsimeq t_1'$, $t_2 \backsimeq t_2'$ and $t = t_1 + t_2$ and $t' = t_1' + t_2'$. An ordered point triad t is not said to be equivalent to an ordered point triad t' unless it follows, by a finite number of applications of the criteria (1), (2), (3), (4), that $t \backsimeq t'$.

Since any translation is a product of two point reflections, Criteria (1) and (3) give

THEOREM 32. *Two ordered point triads are equivalent if they are conjugate under the group of translations and point reflections.*

THEOREM 33. *If A, B, and C are noncollinear points, $ABC \backsimeq ABC$, $ABC \backsimeq BCA$, $ABC \backsimeq CAB$.*

Proof. From (2) of the definition it follows that $ABC \backsimeq BCA$ and $BCA \backsimeq CAB$. Hence, by (3), $ABC \backsimeq CAB$. But, by (2), $CAB \backsimeq ABC$. Hence, by (3), $ABC \backsimeq ABC$.

From the last two theorems and from the form of the definition we now have at once

THEOREM 34. *If $t_1 \backsimeq t_2$, then $t_2 \backsimeq t_1$.*

THEOREM 35. *If A, B, C are any three noncollinear points and O the mid-point of the pair AB, then $AOC \backsimeq OBC$.*

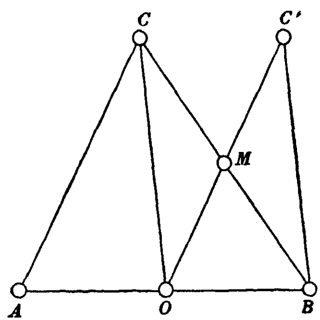


FIG. 34

Proof. Let C' be the point to which C is changed by the translation shifting A to O, and let M be the point of intersection of the non-parallel lines BC and OC'. Since $COBC'$ is a parallelogram, M is the mid-point of the pairs CB and $C'O$. Thus we have

$$AOC \backsimeq OBC' \backsimeq BC'O = BC'M + BMO$$

and
$$OBC = OBM + OMC.$$

But the point reflection with M as center carries OMC into $C'MB$. Thus

$$OMC \backsimeq C'MB \backsimeq BC'M,$$

and
$$OBM \backsimeq BMO,$$

and hence, by comparison with the equivalences and equations above,

$$AOC \backsimeq OBC.$$

THEOREM 36. *Two ordered point triads ABC_1 and ABC_2, where $C_1 \neq C_2$, are equivalent if the line C_1C_2 is parallel to the line AB.*

Proof. Let C_3 be such that B is the mid-point of C_1C_3, and let the line C_2C_3 meet the line AB in O, which is an ordinary point because C_3 is not on the line C_1C_2. It follows (§ 40) that O is the mid-point of the pair C_2C_3.

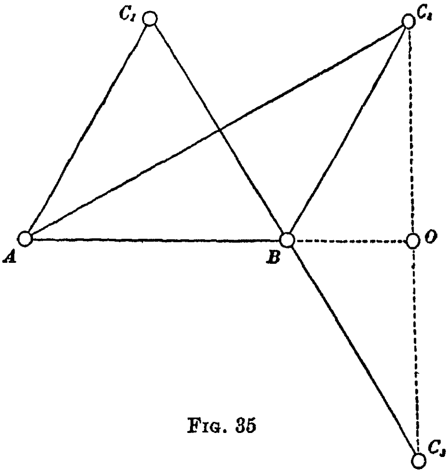


FIG. 35

By Theorems 34 and 35, $ABC_1 \backsimeq BAC_3 \backsimeq C_3BA$. By definition, $C_3BA = C_3BO + C_3OA$. By Theorem 35, $C_3BO \backsimeq C_2OB$ and $C_3OA \backsimeq C_2AO$. Hence $C_3BA \backsimeq C_2AO + C_2OB = C_2AB \backsimeq ABC_2$. Hence $ABC_1 \backsimeq ABC_2$.

COROLLARY. *If a point B' is on a line OB and a point C' on a different line OC, and the lines BC' and $B'C$ are parallel, $BOC \backsimeq B'OC'$.*

Proof. By hypothesis,

$$BOC = BOC' + BC'C$$

and $C'B'O = C'B'B + C'BO$.

But $C'B'B \backsimeq C'CB \backsimeq BC'C$, by Theorems 36 and 34, and $C'BO \backsimeq BOC'$, by Theorem 34. Hence $BOC \backsimeq C'B'O \backsimeq B'OC'$.

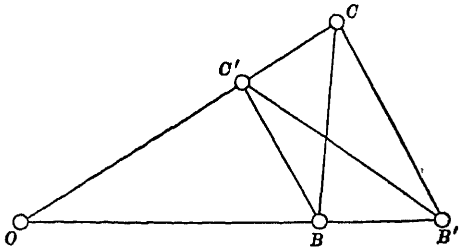


FIG. 36

THEOREM 37. *If A, B, and C are any three noncollinear points, and P and Q are any two distinct points, there exists a line r parallel to PQ such that if R is any point of r, $ABC \backsimeq PQR$.*

Proof. Let T be the translation such that $\mathrm{T}(A) = P$, and let $\mathrm{T}(B) = B'$ and $\mathrm{T}(C) = C'$. If B' is not on the line PQ, let R' be the intersection (fig. 37) of the line through C' parallel to PB' with the line through P parallel to QB'. If B' is on the line PQ, let R' be the point of intersection with PC' of the line through B' parallel to QC'.

In both cases the lines which intersect in R' are by hypothesis non-parallel, so that R' is always an ordinary point. By Theorem 32. $ABC \leftrightharpoons PB'C'$. In case B' is not on PQ, it follows, by Theorem 36, that $PB'C' \leftrightharpoons PB'R' \leftrightharpoons PQR'$. In case B' is on PQ, it follows, by the corollary of Theorem 36, that $PB'C' \leftrightharpoons PQR'$. By Theorem 36 the line r through R' parallel to PQ is such that for every point R on r, $ABC \leftrightharpoons PQR$.

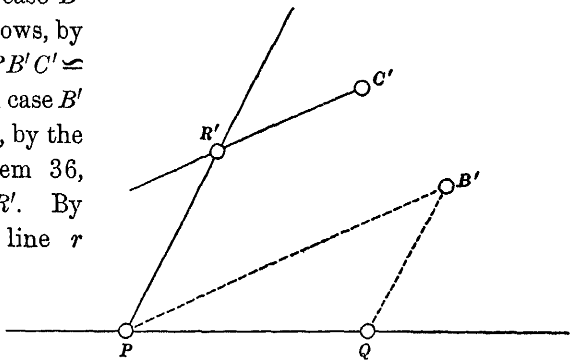


FIG. 37

EXERCISES

1. Two ordered point triads ABC and $AB'C'$ are equivalent if the points B, C, B', C' are collinear and $\text{Vect}(BC) = \text{Vect}(B'C')$.

2. Let O be the point of intersection of the asymptotes l and m of a hyperbola, and let L and M be the intersections with l and m respectively of a variable tangent to the hyperbola. Then the ordered point triads OLM are all equivalent.

49. Measure of ordered point triads. The theorems of the last section state sufficient conditions for the equivalence of ordered point triads. In order to obtain necessary conditions, we shall introduce the notion of *measure*, analogous to the magnitude of a vector.

DEFINITION. Let O, P, Q be three non-collinear points. The *measure* of an ordered point triad ABC *relative to the ordered triad OPQ as a unit* is a number $m(ABC)$ determined as follows: If the line BC is not parallel to OP, let B_1 and C_1

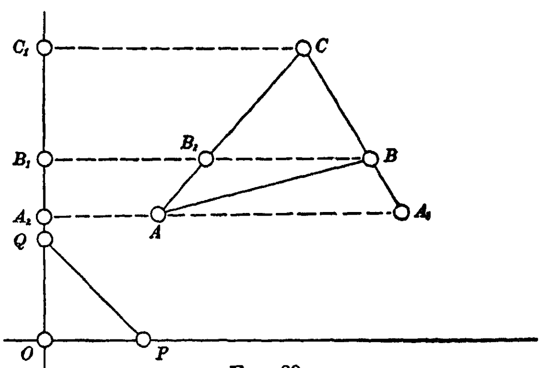


FIG. 38

to OP, meet the line OQ, and let A_1 be the point in which the line through A, parallel to OP, meets the line BC. Let AA_1 denote the magnitude of the vector AA_1 relative to the unit OP (§ 43), and B_1C_1 the magnitude of the vector B_1C_1 relative to the unit OQ. The measure of the ordered triad ABC is*

$$AA_1 \cdot B_1C_1$$

and is denoted by $m(ABC)$. If the line BC is parallel to OP, CA is not parallel to OP, and the measure of ABC is defined to be $m(BCA)$.

If this definition be allowed to apply to any ordered point triad whatever (instead of only to noncollinear triads, cf. § 48), we have $m(ABC) = 0$ whenever the points A, B, C are collinear.

THEOREM 38. *If* $ABC \backsimeq A'B'C'$, *then* $m(ABC) = m(A'B'C')$.

Proof. Let us examine the four criteria in the definition of equivalence in § 48.

(1) In case ABC is carried to $A'B'C'$ by a point reflection, each of the vectors AA_1 and B_1C_1 is transformed into its negative (Theorem 30, corollary), and hence the product of their magnitudes is unchanged.

(2) According to the second criterion, $ABC \backsimeq BCA$. Suppose, first, that neither BC nor CA is parallel to OP, and let A_1, B_1, C_1 have the meaning given them in the definition above. Then

$$m(ABC) = AA_1 \cdot B_1C_1.$$

Let B_2 (fig. 38) be the point in which the line through B, parallel to OP, meets the line CA, and let A_2 be the point in which OQ is met by the parallel to OP through A. Then if BB_2 and C_1A_2 represent the magnitudes of the corresponding vectors relative to OP and OQ as units,

$$m(BCA) = BB_2 \cdot C_1A_2.$$

By Theorem 20,

$$\frac{AA_1}{B_2B} = \frac{A_1C}{BC}.$$

But since the lines CC_1, A_1A_2, BB_1 are parallel, it follows from § 43 that

$$\frac{A_1C}{BC} = \frac{A_2C_1}{B_1C_1}.$$

Hence

$$\frac{AA_1}{B_2B} = \frac{A_2C_1}{B_1C_1},$$

or

$$m(ABC) = AA_1 \cdot B_1C_1 = BB_2 \cdot C_1A_2 = m(BCA).$$

* The factor $\frac{1}{2}$ is lacking in this expression, because we are taking a triangle

In case BC is parallel to OP, the last clause of the definition states that

$$m(ABC) = m(BCA).$$

In case CA is parallel to OP, AB and BC are not parallel to OP, and hence the argument above shows that

$$m(CAB) = m(ABC).$$

But, by definition, $\quad m(BCA) = m(CAB).$

Hence $\quad m(ABC) = m(BCA).$

(3) Corresponding to the fact that if $t_1 \backsimeq t_2$ and $t_2 \backsimeq t_3$, then $t_1 \backsimeq t_3$, we have that, since $m(t)$ is a uniquely defined number, if $m(t_1) = m(t_2)$, and $m(t_2) = m(t_3)$, then $m(t_1) = m(t_3)$.

(4) Let B, C, D be three collinear points and A any point not on the line BC (fig. 39). In case the line BC is not parallel to OP, let A_1 be the point in which the line through A, parallel to OP, meets BC, and let B_1, C_1, D_1 be the points in which the lines through B, C, D respectively, parallel to OP, meet OQ. Then

$$\begin{aligned} m(ABD) &= AA_1 \cdot B_1D_1 \\ &= AA_1 \cdot B_1C_1 + AA_1 \cdot C_1D_1 \\ &= m(ABC) + m(ACD). \end{aligned}$$

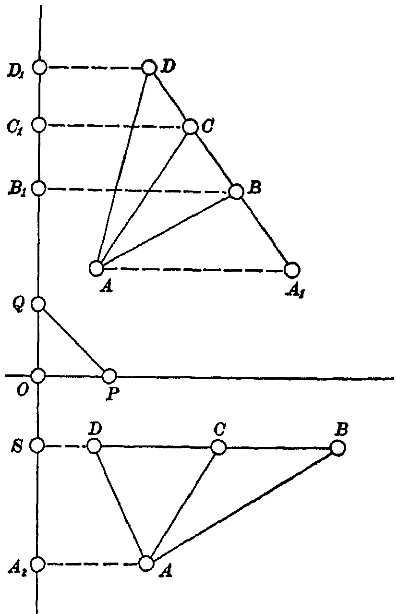


FIG. 39

In case the line BC is parallel to OP, let S be the point in which BC meets OQ, and A_2 be the point in which the line through A, parallel to OP, meets OQ. Then

$$\begin{aligned} m(ABD) = m(BDA) &= BD \cdot SA_2 = BC \cdot SA_2 + CD \cdot SA_2 \\ &= m(BCA) + m(CDA) = m(ABC) + m(ACD). \end{aligned}$$

Thus, in every case, if $t_1 + t_2 = t_3$, $m(t_1) + m(t_2) = m(t_3)$.

Comparing the results proved in these four cases with the definition of equivalence, we have at once that whenever $t_1 \backsimeq t_2$, $m(t_1) = m(t_2)$.

THEOREM 39. *If B, C, and D are collinear points, and the point A is not on the line BC,*

$$\frac{m(ABC)}{m(ABD)} = \frac{BC}{BD}.$$

Proof. In case the line BC is not parallel to OP, let A_1, B_1, C_1 have the meaning given them in the definition of measure, and let D_1 be the point in which the line through D, parallel to OP, meets OQ (fig. 39). Then

$$\frac{m(ABC)}{m(ABD)} = \frac{AA_1 \cdot B_1C_1}{AA_1 \cdot B_1D_1} = \frac{B_1C_1}{B_1D_1}.$$

But, by § 43,

$$\frac{B_1C_1}{B_1D_1} = \frac{BC}{BD}.$$

In case BC is parallel to OP, let A_2 be the point in which the line through A, parallel to OP, meets OQ, and S the point in which BC meets OQ. Then

$$\frac{m(ABC)}{m(ABD)} = \frac{m(BCA)}{m(BDA)} = \frac{BC \cdot SA_2}{BD \cdot SA_2} = \frac{BC}{BD}.$$

COROLLARY 1. *If B, C, D, E are points no two of which are collinear with a point A,*

$$R(AB, AC, AD, AE) = \frac{m(ABD)}{m(ABE)} \div \frac{m(ACD)}{m(ACE)}.$$

COROLLARY 2. *If B, C, D are points no two of which are collinear with a point A, and if P_∞ is the point at infinity of the line CD (the latter not being parallel to AB),*

$$R(AP_\infty, AB, AC, AD) = \frac{m(ABD)}{m(ABC)}.$$

THEOREM 40. *If $m(ABC) = m(A'B'C') \neq 0$, then* $ABC \backsimeq A'B'C'$.

Proof. By Theorem 37 there exists a point C'' on the line $A'C'$ such that $ABC \backsimeq A'B'C''$. Hence $A'B'C' \backsimeq A'B'C''$, and by the last theorem, $C' = C''$.

In consequence of the last two theorems the unit point triad may

Proof. The unit triad OPQ may be chosen so that OP is parallel to AB. Then if C_1 is the point in which the line through C, parallel to OP, meets OQ, and B_1 the point in which AB meets OQ,

$$m(ABC) = AB \cdot B_1C_1.$$

If C_1' is the point in which the line through C', parallel to OP, meets OQ,

$$m(ABC') = AB \cdot B_1C_1'.$$

By Theorem 38, $m(ABC) = m(ABC')$, and hence $C_1 = C_1'$. Hence the line CC' is parallel to AB.

THEOREM 42. *If $ABC \backsimeq AB'C'$, and B' is on the line AB, and C' on the line AC, then the line BC' is parallel to the line $B'C$.*

Proof. By the corollary of Theorem 36, if C'' is a point of AC' such that BC'' is parallel to $B'C$, then

$$ABC \backsimeq AB'C''.$$

By Theorem 41 the only points $\overline{C}$ such that $ABC \backsimeq AB'\overline{C}$ are on the line through C'', parallel to AB'. Hence $C' = C''$.

It is notable that although the sufficient conditions for equivalence given in § 48 are all proved on the basis of Assumptions A, E, H_0, the discussion of the ratios of vectors, and hence all the necessary conditions for equivalence, involve Assumption P in their proofs. This is essential,* as we can show by proving that Assumption P is a logical consequence of these theorems, together with the previous theorems on equivalence. As was pointed out in § 3, Assumption P is a logical consequence of the theorem of Pappus, Theorem 21, § 36, Vol. I. When one of the lines of the configuration is taken as l_∞, this theorem assumes the form:

If a simple hexagon $AB'CA'BC'$ is such that A, B, C are on one line and A', B', C' on another line, and if AB' is parallel to $A'B$ and BC' parallel to $B'C$, then CA' is parallel to $C'A$.

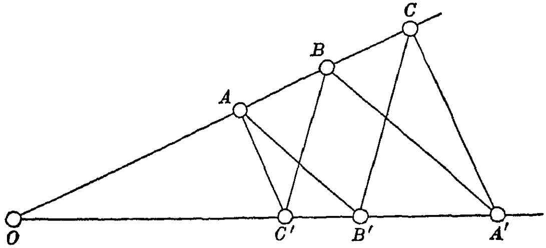


FIG. 40

In case the lines containing ABC and $A'B'C'$, respectively, are parallel, this can be proved from the Desargues theorem on perspective triangles; so that we are interested only in the

* The rôle of Assumption P (or rather of the equivalent theorem of Pappus) in the theory of areas was first determined in a definite way by D. Hilbert, Grundlagen

projective geometry if it be desired to base projective geometry upon elementary Euclidean geometry (cf. Ex. 3, § 54).

The notion of measure can be extended to any ordered set of n points, i.e. (cf. § 14, Vol. I) to any *simple n-point.* The details of this discussion are left to the reader. An outline is furnished by the problems below. The principal references are to A. F. Möbius, Der barycentrische Calcul, §§ 1, 17, 18, 165; Werke, Vol. I, pp. 23, 39, 200; Vol. II, p. 485. See also the Encyclopädie der Math. Wiss., III AB 9, § 12. It is to be borne in mind in using these references that our hypotheses are narrower than those used by the previous writers.

EXERCISES

1. For any three points A, B, C,

$$m(ABC) + m(ACB) = 0.$$

2. For any four points O, A, B, C,

$$m(ABC) = m(OAB) + m(OBC) + m(OCA).$$

3. For any n points $A_1, A_2, \cdots, A_n$ the number

$$m(OA_1A_2) + m(OA_2A_3) + \cdots + m(OA_{n-1}A_n) + m(OA_nA_1)$$

is the same for all choices of the point O. We define it to be the *measure* of the simple n-point $A_1A_2 \cdots A_n$ and denote it by $m(A_1A_2 \cdots A_n)$.

4. $m(A_1A_2 \cdots A_{n-1}A_n) = m(A_2A_3 \cdots A_nA_1)$.

5. $m(A_1A_2 \cdots A_n) + m(A_1A_nA_{n+1} \cdots A_{n+k}) = m(A_1A_2 \cdots A_{n+k})$.

6. Derive a formula for $m(A_1A_2 \cdots A_n)$ analogous to the definition of $m(ABC)$ in terms of vectors collinear with two arbitrary vectors OP and OQ.

7. Prove the converse propositions to those stated in the exercises in § 48.

8. If $ABCD$ and $A'B'C'D'$ are two parallelograms whose sides are respectively parallel,

$$\frac{m(ABCD)}{m(A'B'C'D')} = \frac{AB}{A'B'} \cdot \frac{BC}{B'C'}.$$

9. The variable parallelogram two of whose sides are the asymptotes of a hyperbola and one vertex of which is on the hyperbola has a constant measure.

10. If a variable pair of conjugate diameters meets a conic in point pairs AA', BB', the parallelogram whose sides are the tangents at A, A', B, B' has a constant measure. The parallelogram $ABA'B'$ also has a constant measure.

50. The equiaffine group. THEOREM 43. *If two equivalent ordered point triads* t_1 *and* t_2 *are transformed by an affine collineation into* t_1' *and* t_2', then $t_1' \backsimeq t_2'$.

Proof. It is necessary merely to verify that the relation used in each of the criteria (1), $\cdots$, (4) in the definition of equivalence (§ 48) is unaffected by an affine collineation. For Criterion (1) this reduces to Theorem 28. For Criteria (2), (3), (4) it is a consequence of the fact that an affine collineation transforms ordered triads into ordered triads and collinear points into collinear points.

THEOREM 44. *If an affine collineation transforms one ordered point triad into an equivalent point triad, it transforms every ordered point triad into an equivalent point triad.*

Proof. It follows from Theorem 43 that if ABC is transformed by a given collineation into an equivalent ordered point triad $A'B'C'$, then every point triad equivalent to ABC is transformed into a point triad equivalent to $A'B'C'$ and thus into one equivalent to ABC. By Theorem 37 any ordered point triad whatever is equivalent to some point triad ADC, where D is on the line AB. Hence the present theorem will be proved if we can show that ADC is transformed into an equivalent point triad.

Denote the point to which D is transformed by the given collineation by D'. By Theorem 39,

$$\frac{m(ADC)}{m(ABC)} = \frac{AD}{AB} \text{ and } \frac{m(A'D'C')}{m(A'B'C')} = \frac{A'D'}{A'B'}.$$

By § 43,

$$\frac{AD}{AB} = \text{℞}(P_\infty A, BD),$$

where P_∞ is the point at infinity of the line AB. But since the given collineation is affine, P_∞ is transformed to the point at infinity P_∞' of the line $A'B'$, and

$$\text{℞}(P_\infty A, BD) = \text{℞}(P_\infty' A', B'D') = \frac{A'D'}{A'B'} = \frac{m(A'D'C')}{m(A'B'C')}.$$

Since $m(ABC) = m(A'B'C')$, it follows that $m(ADC) = m(A'D'C')$. Hence

$$ADC \backsimeq A'D'C'.$$

DEFINITION. Any affine collineation which transforms an ordered

subgroup of the affine group.

Proof. By the last theorem an equiaffine collineation transforms every ordered point triad into an equivalent point triad. Hence, by Condition (3) in the definition of equivalence, the product of two equiaffine collineations is equiaffine. By Theorem 43, $\Sigma T \Sigma^{-1}$ is equiaffine whenever T is equiaffine and Σ affine.

THEOREM 46. *Let A, B, A', B' be points such that $A \neq B$ and $A' \neq B'$; let a be a line on A but not on B, and let a' be a line on A' but not on B'. There is one and only one equiaffine collineation transforming A to A', B to B', and a to a'.*

Proof. Let C be any point distinct from A on a. By Theorem 37, there is a point C' on the line a' such that

$$ABC \backsimeq A'B'C'.$$

By Theorem 1 there is one and only one affine transformation carrying A, B, C to A', B', C' respectively, and by definition this transformation is equiaffine. By Theorem 41, C' is the only point on a' such that $ABC \backsimeq A'B'C'$. Hence (Theorem 44) there is only one equiaffine transformation carrying A, B, a into A', B', a' respectively.

EXERCISE

Any affine collineation leaves invariant the ratio of the measures of any two point triads.

***51. Algebraic formula for measure. Barycentric coördinates.** Consider a nonhomogeneous coördinate system in which l_∞ is the singular line. Let the unit of measure for ordered triads be OPQ, where $O = (0, 0)$, $P = (1, 0)$, $Q = (0, 1)$. Let $A = (a_1, a_2)$, $B = (b_1, b_2)$, $C = (c_1, c_2)$; the line through A, parallel to OP, consists of the points $(a_1 + \lambda, a_2)$, where λ is arbitrary, and the line BC has the equation (§ 64, Vol. I),

$$\begin{vmatrix} x & y & 1 \\ b_1 & b_2 & 1 \\ c_1 & c_2 & 1 \end{vmatrix} = 0.$$

In case the line BC is not parallel to OP, and therefore $b_2 \neq c_2$, the point A_1 which appears in the definition of measure (§ 49) is $(a_1 + \lambda, a_2)$, where λ satisfies

$$\begin{vmatrix} \lambda & 0 & 0 \\ b_1 & b_2 & 1 \\ c_1 & c_2 & 1 \end{vmatrix} + \begin{vmatrix} a_1 & a_2 & 1 \\ b_1 & b_2 & 1 \\ c_1 & c_2 & 1 \end{vmatrix} = 0.$$

Hence
$$AA_1 = \frac{-1}{(b_2 - c_2)} \cdot \begin{vmatrix} a_1 & a_2 & 1 \\ b_1 & b_2 & 1 \\ c_1 & c_2 & 1 \end{vmatrix}.$$

The points B_1 and C_1 of the definition of measure are $(0, b_2)$ and $(0, c_2)$, respectively, so that
$$B_1C_1 = c_2 - b_2.$$

Hence
$$(10) \qquad m(ABC) = \begin{vmatrix} a_1 & a_2 & 1 \\ b_1 & b_2 & 1 \\ c_1 & c_2 & 1 \end{vmatrix}.$$

That the same result holds good in case BC is parallel to OP is readily verified.

Now if A, B, C are transformed to A', B', C' respectively by a transformation
$$(11) \qquad \begin{aligned} x' &= \alpha_1 x + \beta_1 y + \gamma_1, \\ y' &= \alpha_2 x + \beta_2 y + \gamma_2, \end{aligned} \quad \Delta = \begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix} \neq 0$$

of the affine group,
$$(12) \qquad m(A'B'C') = \begin{vmatrix} \alpha_1 a_1 + \beta_1 a_2 + \gamma_1, & \alpha_2 a_1 + \beta_2 a_2 + \gamma_2, & 1 \\ \alpha_1 b_1 + \beta_1 b_2 + \gamma_1, & \alpha_2 b_1 + \beta_2 b_2 + \gamma_2, & 1 \\ \alpha_1 c_1 + \beta_1 c_2 + \gamma_1, & \alpha_2 c_1 + \beta_2 c_2 + \gamma_2, & 1 \end{vmatrix}$$
$$= \begin{vmatrix} a_1 & a_2 & 1 \\ b_1 & b_2 & 1 \\ c_1 & c_2 & 1 \end{vmatrix} \cdot \begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix}.$$

Hence we have

THEOREM 47. *A transformation* (11) *of the affine group is equiaffine if and only if**
$$\begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix} = 1.$$

Let $A = (a_1, a_2)$, $B = (b_1, b_2)$, $C = (c_1, c_2)$ be the vertices of any triangle, and $P = (x, y)$ any point. In the homogeneous coördinates for which $x_1/x_0 = x$, $x_2/x_0 = y$, these points may be written $A = (1, a_1, a_2)$, etc. Hence by the result established in § 27 for the three-dimensional case, the numbers proportional to
$$\xi_0 = \begin{vmatrix} 1 & x & y \\ 1 & b_1 & b_2 \\ 1 & c_1 & c_2 \end{vmatrix}, \quad \xi_1 = \begin{vmatrix} 1 & a_1 & a_2 \\ 1 & x & y \\ 1 & c_1 & c_2 \end{vmatrix}, \quad \xi_2 = \begin{vmatrix} 1 & a_1 & a_2 \\ 1 & b_1 & b_2 \\ 1 & x & y \end{vmatrix}$$

may be regarded as homogeneous coördinates of P in a system for which ABC is the triangle of reference.

an equiaffine collineation. In like manner, $\{L_2l_2\}\cdot\{Ll\}$ is also equiaffi
Hence the product $\{L_2l_2\}\cdot\{L_1l_1\}$ is equiaffine.

THEOREM 49. *An equiaffine collineation is a product of two l reflections.*

Proof. Let Γ be any equiaffine collineation. If there be any po which is not on an invariant line of Γ, let A_1 be such a point. A_0, A_2, A_3 be defined by the conditions

$$\Gamma(A_0)=A_1,\ \ \Gamma(A_1)=A_2,\ \ \Gamma(A_2)=A_3.$$

By the hypothesis on A_1 the points A_0, A_1, A_2 are noncollinear, and the hypothesis that Γ is equiaffine

$$A_0A_1A_2 \simeq A_1A_2A_3 \simeq A_3A_1A_2.$$

Hence, by Theorem 41, the line A_0A_3 is parallel to A_1A_2, or else $A_0=$

Let M_1 be the mid-point of the pair A_0A_2, and M_2 of the pair A
Let L_1 be the point at infinity of the line A_0A_2, L_2 of the line A_1A_2, a K of the line A_1A_3. Since A_0A_3 is parallel to A_1A_2, it follows t

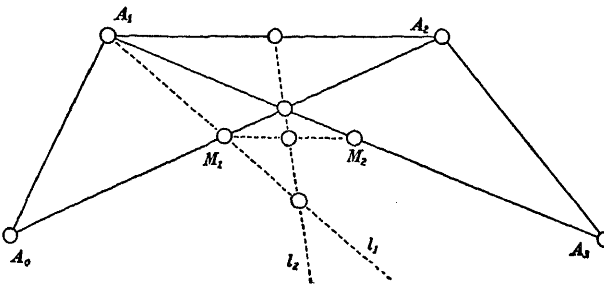


FIG. 42

$A_0A_2L_1 \overset{L_2}{\underset{\wedge}{=}} A_3A_1K$, and hence, by the definition of mid-point, that M_1, and L_2 are collinear. Since A_0, A_2, and the point at infinity of the l A_0A_2 are transformed by Γ to A_1, A_3, and the point at infinity of line A_1A_3, $\Gamma(M_1)=M_2$.

Let l_1 be the line A_1M_1, and l_2 the line joining the mid-point of A to the mid-point of M_1M_2. By the above,

$$\{L_1l_1\}(A_0A_1M_1)=A_2A_1M_1,$$

and
$$\{L_2l_2\}(A_2A_1M_1)=A_1A_2M_2.$$

Hence
$$\{L_2l_2\}\cdot\{L_1l_1\}(A_0A_1M_1)=A_1A_2M_2.$$

But since $\Gamma(A_0A_1M_1)=A_1A_2M_2$, it follows, by Theorem 1, that

$$\Gamma=\{L_2l_2\}\cdot\{L_1l_1\}.$$

at least two ordinary points in common. Thus we should be led to a contradiction with Theorem 46 if the invariant lines were not concurrent.

Let A_1 be a point which is not invariant, and let $A_2 = \Gamma(A_1)$. Also let B_1 be another point which is not invariant and not on the line A_1A_2, and let $\Gamma(B_1) = B_2$. The lines A_1A_2 and B_1B_2 necessarily meet in O.

If O is ordinary, then since any line through it is invariant, all points of l_∞ are invariant, and hence A_1B_1 is parallel to A_2B_2. Since Γ is equiaffine,

$$A_1B_1O \backsimeq A_2B_2O.$$

Hence, by Theorem 42, A_1B_2 and A_2B_1 are parallel, and $A_1B_1A_2B_2$ is a parallelogram. Hence O is the mid-point of A_1A_2 and B_1B_2, and Γ is a point reflection.

Let a be the line A_1A_2 and A the point at infinity of a, and let b be the line B_1B_2 and B the point at infinity of b. The product $\{Ab\} \cdot \{Ba\}$ transforms A_1, B_1, O into A_2, B_2, O respectively, and hence is Γ.

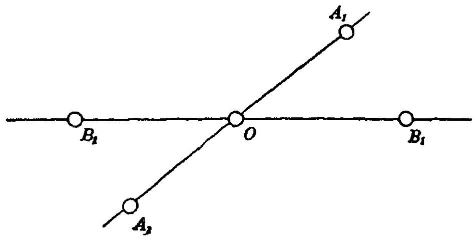


FIG. 43

If O is an ideal point, let l be the line A_1B_1, and let m be the line joining the mid-points of A_1A_2 and B_1B_2. Then $\{Om\} \cdot \{Ol\}$ transforms O, A_1, B_1 into O, A_2, B_2 respectively, and hence, by Theorem 46, is Γ.

COROLLARY 1. *An equiaffine collineation* Γ *such that* A, $\Gamma(A)$ *and* $\Gamma^2(A)$ *are collinear for all choices of* A *is either a point reflection or a translation or an elation whose center is at infinity and whose axis is an ordinary line.*

Proof. In the argument above it was proved that if the point O is ordinary, Γ is a point reflection; and that if O is ideal, $\Gamma = \{Om\} \cdot \{Ol\}$. If m and l are parallel, Γ is evidently a translation; and if m and l are not parallel, it is an elation with O as center and the line joining O to the point lm as axis.

DEFINITION. An elation whose center is at infinity and whose axis is an ordinary line is called a *simple shear.*

COROLLARY 2. *If* $\Gamma = \{L_2 l_2\} \cdot \{L_1 l_1\}$, *then for every line* l *concurrent with* l_1 *and* l_2 *which is not a double line of* Γ *there exist points* L *and* M *and a line* m *such that*

$$\Gamma = \{Mm\} \cdot \{Ll\}.$$

There also exist a point M' *and a line* m' *such that*

$$\Gamma = \{Ll\} \cdot \{M'm'\}.$$

If l *be taken as variable,*

$$[l] \,\overline{\wedge}\, [L] \,\overline{\wedge}\, [m] \,\overline{\wedge}\, [M] \,\overline{\wedge}\, [M'] \,\overline{\wedge}\, [m'].$$

Proof. The first conclusion follows from the arbitrariness in the choice of A_1 in the proof of the theorem above. The second conclusion follows from the first, combined with the fact that

$$\Gamma^{-1} = \{L_1 l_1\} \cdot \{L_2 l_2\}.$$

The projectivities follow from the constructions given in the proof of theorem for A_0, A_2, M_1, etc.

COROLLARY 3. *If* $\Gamma = \{L_2 l_2\} \cdot \{L_1 l_1\}$, *then for every point* L *of* l_∞ *which is not a double point of* Γ, *there exists a point* M *of* l_∞ *and two lines* l *and* m *concurrent with* l_1 *and* l_2 *such that*

$$\Gamma = \{Mm\} \cdot \{Ll\}.$$

There also exist a point M' *and a line* m' *such that*

$$\Gamma = \{Ll\} \cdot \{M'm'\}.$$

THEOREM 50. *The set of all affine collineations which are products of line reflections form a group. Every transformation of this group is either an equiaffine transformation or the product of an equiaffine transformation by a line reflection.*

Proof. By Theorems 48 and 49 the product of an even number of line reflections is equiaffine and reduces to a product of two line reflections. Hence the product of an odd number of line reflections reduces to a product of three line reflections. The statements above

EXERCISES

1. Let the points at infinity of l_1, l_2, l respectively in Theorem 49, Cor. 2, be denoted by L_1', L_2', L'. If the points L_1, L_1', L_2, L_2' are distinct, the pairs L_1L_1', L_2L_2', LL' are in involution.

2. In case L_1 is on l_2 and L_2 is not on l_1, $\{L_2l_2\} \cdot \{L_1l_1\} = \mathrm{T}$ is a collineation of Type *II* (cf. § 40, Vol. I), parabolic on l_∞ and of period two on the line joining L_1 to the point of intersection of l_1 and l_2. If l be any line, except l_2, through the point l_1l_2, P the point in which l meets l_∞, and L the harmonic conjugate of L_1 with respect to P and $\mathrm{T}(P)$,

$$\mathrm{T} = \{Ll_2\} \cdot \{L_1l\}.$$

If M be the harmonic conjugate of L_1 with respect to P and $\mathrm{T}^{-1}(P)$, $\mathrm{T} = \{L_1l\} \cdot \{Ml_2\}$.

3. The product $\{L_2l_2\} \cdot \{L_1l_1\}$ is a point reflection if and only if L_1 is on l_2 and L_2 on l_1. A point reflection with O as center is the product of any two line reflections $\{L_1l_1\}$ and $\{L_2l_2\}$ for which l_1 is on O, l_2 on O, L_1 on l_2, and L_2 on l_1.

4. The product $\{L_2l_2\} \cdot \{L_1l_1\}$ is a translation if and only if $L_1 = L_2$ and l_1 is parallel to l_2. The ideal point L_1 is the center of the translation. If T is any translation, T_∞ its center, P_1 any ordinary point, $P = \mathrm{T}(P_1)$, P_2 the mid-point of the pair PP_1, and p_1 and p_2 two parallel lines through P_1 and P_2 respectively, $\mathrm{T} = \{T_\infty, p_2\} \cdot \{T_\infty, p_1\}$.

5. The product $\{L_2l_2\} \cdot \{L_1l_1\}$ is a simple shear if $L_1 \neq L_2$ and $l_1 = l_2$, or if $L_1 = L_2$ and l_1 intersects l_2 in an ordinary point, but not in any other case.

6. Let Σ be a simple shear whose axis is l and whose center is L. Let P_1 be any point of l_∞, $P = \Sigma(P_1)$, and P_2 the harmonic conjugate of L with respect to P and P_2. Then $\Sigma = \{P_2l\} \cdot \{P_1l\}$. If p_1 be any line meeting l in an ordinary point, $p = \Sigma(p_1)$, and p_2 the harmonic conjugate of l with respect to p and p_1,

$$\Sigma = \{Lp_2\} \cdot \{Lp_1\}.$$

7. Let $PP_1P_2P_3P_4$ be a simple pentagon. Let C_1, C_2, C_3, C_4, C_5 be the mid-points of the pairs $PP_1, P_1P_2, P_2P_3, P_3P_4, P_4P$ respectively. If the line PP_1 is parallel to P_3P_4, and PP_4 is parallel to P_1P_2, the three lines C_1C_4, C_2C_5, PC_3 are concurrent or parallel. Discuss the degenerate cases.

8. Every equiaffine transformation is either the identity or a point reflection or an elation whose center is at infinity (i.e. a translation or a simple shear) or expressible as a product of two elations whose centers are at infinity.

9. Prove Cors. 2 and 3 of Theorem 49 directly, without using the theory of equivalence.

10. A necessary and sufficient condition that a planar collineation be the product of two harmonic homologies is that it transform ordered point triads into equivalent point triads relative to a fixed line of the collineation regarded as l_∞ (E. B. Wilson, Annals of Mathematics, Vol. V, 2d series (1903), p. 45).

11. Let us denote an involution whose double points are L and M by $\{LM\}$. If $I_1=\{L_1M_1\}$ and $I_2=\{L_2M_2\}$ are two distinct involutions on the same line, then for every point L_3 of this line, L_3 not being a double point of $I_1\cdot I_2$, there exists a unique point M_3 and involution $\{L_4M_4\}$ such that if we denote $\{L_3M_3\}$ by I_3 and $\{L_4M_4\}$ by I_4,

$$I_3I_2I_1=I_4, \quad \text{and} \quad I_2I_1=I_3I_4.$$

The pairs L_1M_1, L_2M_2, L_3M_3, L_4M_4 are all pairs of the same involution, unless the pairs L_1M_1 and L_2M_2 have a point in common, in which case all four pairs have this point in common.

12. The projectivities on a line which are expressible in the form $\{L_1M_1\}\cdot\{L_2M_2\}$ form a group.

The last two exercises connect with the following algebraic considerations. An involution in a net of rationality is always of the form (§ 54, Vol. I)

$$x'=\frac{ax+b}{cx-a},$$

where a, b, c, d are rational. The double points are the roots of

$$cx^2-2\,ax-b=0,$$

and both will be rational if k is rational in

$$a^2+bc=k^2.$$

Now any projectivity is the product of two involutions, a double point of one of which may be chosen arbitrarily. The projectivity may therefore be written

$$x''=\frac{a'\dfrac{ax+b}{cx-a}+b'}{c'\dfrac{ax+b}{cx-a}-a'}=\frac{(aa'+b'c)x+(a'b-ab')}{(ac'-a'c)x+(bc'+aa')},$$

and so has the determinant

$$\begin{aligned}&aa'bc'+a^2a'^2+bb'cc'+b'caa'-(aa'bc'-a^2b'c'-a'^2bc+aa'b'c)\\&=a^2(a'^2+b'c')+bc\,(b'c'+a'^2)=k^2k'^2,\end{aligned}$$

where $k'^2=a'^2+b'c'$. Hence (1) the product of two involutions whose double points have rational coördinates is a projectivity whose determinant is a perfect square; and (2) if the determinant of a projectivity is a perfect square, and one of two involutions of which it is a product has rational double points, then the other has rational double points. Hence there is a subgroup of the group of collineations of a linear net of rationality generated by the involutions with rational double points. This is the group of transformations whose

$$\frac{x_1}{x_0} = x, \qquad \frac{x_2}{x_0} = y.$$

The line l_∞ now has the equation $x_0 = 0$, and the equations (1) of the affine group become

$$(13) \qquad \begin{aligned} x_0' &= x_0, \\ x_1' &= c_1 x_0 + a_1 x_1 + b_1 x_2, \\ x_2' &= c_2 x_0 + a_2 x_1 + b_2 x_2, \end{aligned} \qquad \Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0.$$

On the line l_∞ this effects the transformation

$$\begin{aligned} x_1' &= a_1 x_1 + b_1 x_2, \\ x_2' &= a_2 x_1 + b_2 x_2. \end{aligned}$$

According to § 54, Vol. I, this is an involution if and only if $a_1 = -b_2$. Thus $a_1 = -b_2$ is a necessary condition that (13) represent a line reflection.

The ordinary double points of (13) are given by the following equations, in which we have put $a = a_1 = -b_2$.

$$(14) \qquad \begin{aligned} (a-1)x + b_1 y + c_1 &= 0, \\ a_2 x - (a+1)y + c_2 &= 0. \end{aligned}$$

If (13) is to be a line reflection, it must have a line of fixed points. Hence the two equations (14) must represent a single ordinary line, which requires

$$(15) \qquad 0 = \begin{vmatrix} a-1 & b_1 \\ a_2 & -(a+1) \end{vmatrix} = \begin{vmatrix} a-1 & c_1 \\ a_2 & c_2 \end{vmatrix} = \begin{vmatrix} b_1 & c_1 \\ -(a+1) & c_2 \end{vmatrix}.$$

The first of these conditions is equivalent to $\Delta = -1$.

Since the coefficients of x and y in (14) cannot all vanish, the conditions (15) are also sufficient that (14) represent a single ordinary line. Hence

THEOREM 51. *A transformation of the form*

$$(16) \qquad \begin{aligned} x' &= ax + b_1 y + c_1, \\ y' &= a_2 x - ay + c_2, \end{aligned}$$

is a line reflection if and only if

$$\Delta = \begin{vmatrix} a & b_1 \\ a_2 & -a \end{vmatrix} = -1, \qquad \begin{vmatrix} a-1 & c_1 \\ a_2 & c_2 \end{vmatrix} = \begin{vmatrix} b_1 & c_1 \\ -(a+1) & c_2 \end{vmatrix} = 0.$$

From this it follows that a product of two line reflections is such that $\Delta = 1$, and a product of three line reflections is such that $\Delta = -1$. By Theorems 47 and 49 any transformation for which $\Delta = 1$ is a product of two line reflections. Any transformation T for which $\Delta = -1$, when multiplied by a line reflection Λ yields a transformation Σ for which $\Delta = 1$, i.e. an equiaffine transformation. From $T\Lambda = \Sigma$ follows $T = \Sigma\Lambda$. Hence T is a product of three line reflections. Thus we have (cf. Theorem 47)

THEOREM 52. *The group of affine transformations which are products of line reflections has the equations*

$$\begin{aligned} x' &= a_1x + b_1y + c_1, \\ y' &= a_2x + b_2y + c_2, \end{aligned} \qquad \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2 = 1.$$

EXERCISES

1. The set of all affine transformations which are products of equiaffine transformations by dilations form a group which is a self-conjugate subgroup of the affine group. Its equations are

$$\begin{aligned} x' &= a_1x + b_1y + c_1, \\ y' &= a_2x + b_2y + c_2, \end{aligned} \qquad \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = k^2,$$

where k is any number in the geometric number system.

2. The set of all affine transformations which are products of line reflections and dilations form a group which is self-conjugate under the affine group. Its equations are

$$\begin{aligned} x' &= a_1x + b_1y + c_1, \\ y' &= a_2x + b_2y + c_2, \end{aligned} \qquad \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \pm k^2,$$

where k is any number in the geometric number system.

54. Subgroups of the affine group. We give below a list of the principal subgroups of the affine group which we have considered in this chapter and in § 30 of Chap. II. These are all self-conjugate subgroups. We also include the groups which will be considered in the next chapter in connection with the Euclidean geometry.

The groups are all described by means of the conditions which must be imposed on the coefficients of the equations of the affine group to reduce it to each of the other groups. In some spaces, i.e. when the variables and coefficients are in certain number systems, these groups are not all distinct. However, they are all distinct in

where $$\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0.$$

The principal subgroups connected with the affine geometry are:

$$(2) \qquad \Delta > 0;$$

the transformations satisfying this condition are direct (§ 30).

$$(3) \qquad \Delta = k^2,$$

where k is in the geometric number system (§ 53, Ex. 1).

$$(4) \qquad \Delta = \pm k^2,$$

where k is in the geometric number system (§ 53, Ex. 2).

$$(5) \qquad \Delta^2 = 1;$$

these are products of two or of three line reflections (Theorem 52).

$$(6) \qquad \Delta = 1,$$

the equiaffine group (§ 51).

$$(7) \qquad a_2 = b_1 = 0, \quad a_1 = b_2,$$

the homothetic group (§ 47).

$$(8) \qquad a_2 = b_1 = 0, \quad a_1 = b_2, \quad a_1^2 = 1,$$

the group of translations and point reflections (§ 45).

$$(9) \qquad a_2 = b_1 = 0, \quad a_1 = b_2 = 1,$$

the group of translations (§ 38).

The principal groups connected with the Euclidean geometry are:

$$(10) \qquad a_1^2 + a_2^2 = b_1^2 + b_2^2 \neq 0, \quad a_1 b_1 + a_2 b_2 = 0,$$

the Euclidean group (§§ 55 and 62). Its transformations are called similarity transformations.

$$(11) \qquad a_1^2 + a_2^2 = b_1^2 + b_2^2 \neq 0, \quad a_1 b_1 + a_2 b_2 = 0, \quad \Delta > 0,$$

the direct similarity transformations.

$$(12) \qquad a_1^2 + a_2^2 = b_1^2 + b_2^2 \neq 0, \quad a_1 b_1 + a_2 b_2 = 0, \quad \Delta = k^2,$$

where k is in the geometric number system.

$$(13) \qquad a_1^2 + a_2^2 = b_1^2 + b_2^2 \neq 0, \quad a_1 b_1 + a_2 b_2 = 0, \quad \Delta = \pm k^2,$$

where k is in the geometric number system.

$$(14) \qquad a_1^2 + a_2^2 = 1, \quad a_1 = \pm b_2, \quad a_2 = \mp b_1,$$

the group of displacements and symmetries (§ 62).

$$(15) \qquad a_1^2 + a_2^2 = 1, \quad a_1 = b_2, \quad a_2 = - b_1,$$

the group of displacements.

The relations among these groups may be indicated by the following diagram, in which we have included only those groups which are distinct in case of the real geometry. A dotted line indicates that the lower of the two groups joined is a subgroup of the upper, and a solid line that it is a self-conjugate subgroup.

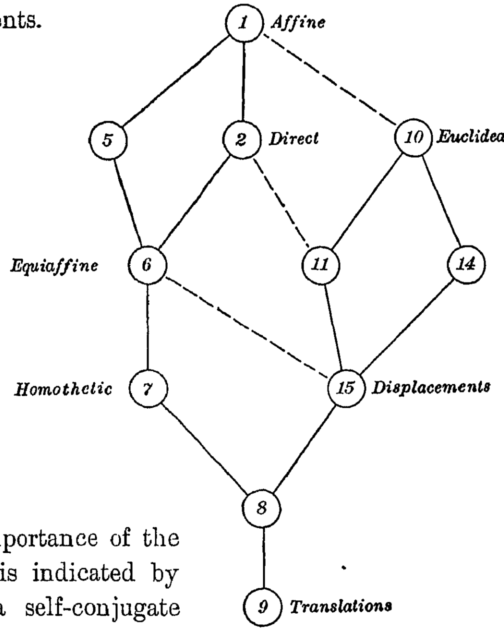


Fig. 44

The fundamental importance of the group of translations is indicated by the fact that it is a self-conjugate subgroup of each of the other groups.

EXERCISES

1. Supposing the number of points on a line to be $p + 1$, what is the numb of transformations in each of the groups listed above?

2. Supposing the geometric number system to be (a) the ordinary real, (b) the ordinary complex number system, how many parameters are there the equations for each of the groups listed above?

3. Prove that the plane affine geometry as a separate science could be bas on the following assumptions with regard to undefined elements, called poir and undefined classes of points, called lines:

I. Two points are contained in one and only one line.

II. For any line l and any point P, not on l, there is one and only one li containing P and not containing any point of l.

III. Every line contains at least two points.

IV. There exist at least three noncollinear points.

V. The special case of the Pappus theorem given in the fine print in § 4

CHAPTER IV

EUCLIDEAN PLANE GEOMETRY

55. Geometries of the Euclidean type. We come now to the extension of the definition of congruence which was promised in §§ 39 and 46. This requires the consideration of groups which are not self-conjugate under the affine group. Not being self-conjugate, these groups are not determined uniquely by the affine group, and hence our definitions will contain a further arbitrary element.

DEFINITION. Let I be an arbitrary but fixed involution on l_∞. This involution shall be called the *absolute* or *orthogonal involution.* The group of all projective collineations leaving I invariant shall be called a *parabolic* * *metric group.* The transformations of the group shall be called *similarity transformations.* Two figures conjugate under the group shall be said to be *similar.* The geometry corresponding to the group shall be called the *parabolic metric geometry.*

The absolute involution is supposed to be fixed throughout the rest of the discussion, but of course there are as many parabolic metric groups as there are choices of I. We nevertheless speak of *the* parabolic metric group in order to emphasize the fact that we are fixing attention on one group.

In case the plane in which we are working is a real plane and the absolute involution is without double points, the parabolic metric geometry is the Euclidean geometry. It is for this reason that we refer to the parabolic metric geometries as geometries of the Euclidean type.

The investigations in the following sections are arranged in order of increasing specialization. First we consider a perfectly general involution, I, in a projective plane satisfying A, E, P, H_0. Then we consider a particular type of involution in an ordered plane, and finally limit the plane to be the real plane.

* The reason for the term "parabolic" in this connection is explained in a later chapter, where the elliptic and hyperbolic metric groups are defined.

plane in which the absolute involution has double points. Thus the theorems on the general type of involution (where the possible existence of double points is taken into account) come to have a new application.

56. Orthogonal lines. DEFINITION. Two lines are said to be *orthogonal* or *perpendicular* to each other if and only if they meet l_∞ in conjugate points of the absolute involution.

The following consequences of this definition are obvious:

THEOREM 1. *The pairs of perpendicular lines through any point, O, are the pairs of an involution. Through any point there is one and but one line perpendicular to a given line. A line perpendicular to one of two parallel lines is perpendicular to the other. Two lines perpendicular to the same line are parallel.*

DEFINITION. In case the absolute involution I has two double points, I_1 and I_2, they are called the *circular points.* Any line through I_1 or I_2 is called an *isotropic line* or a *minimal line.*

Any isotropic line has the property of being perpendicular to itself. The circular points are so called because all ordinary points of any circle (cf. § 60) are on a conic through I_1 and I_2. The ordinary points of the conic section referred to in the following lemma will later be proved to be on a circle.

DEFINITION. A homology of period two whose center L is on l_∞, and whose axis l meets l_∞ in the point conjugate to the center with regard to the absolute involution, is called an *orthogonal line reflection,* and is denoted by $\{Ll\}$.

Since the center of a homology is not a point of the axis, the center cannot be a double point of the orthogonal involution, nor can the axis pass through such a point. An orthogonal line reflection is of course a special case of a line reflection as defined in § 52.

LEMMA. *Let O and P_1 be two points not collinear with either double point of the absolute involution. There is one and only one conic, C^2, having O as center, passing through P_1, and having the pairs of the absolute involution as pairs of conjugate points.*

Proof. Let P_2 be the harmonic conjugate of P_1 with respect to O and the point at infinity, P_∞, of the line OP_1. Any conic containing P_1

and having O as center must contain P_2, by the definition of center. Let X be a variable point of l_∞, and Y the conjugate of X in the absolute involution. Any of the triangles OXY must be self-polar to any conic satisfying the required conditions. But if P is the point of intersection of the lines P_1X and P_2Y, and Q the point of intersection of P_1X and OY,

$$P_2OP_1P_\infty \overset{Y}{\underset{\wedge}{=}} PQP_1X,$$

and hence the points P_1 and P are harmonically conjugate with respect to X and Q. Hence P must be on any conic through P_1 with regard to which X is the pole of QY. Hence P must be on any conic satisfying the hypotheses of the lemma.

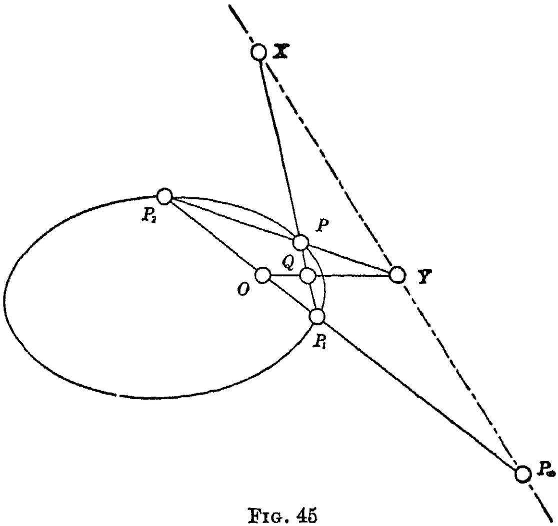


FIG. 45

Since $P_1[X] \overline{\wedge} P_2[Y]$, the points P, together with P_1 and P_2, constitute a unique conic (§ 41, Vol. I); and this conic, by its construction, satisfies the condition required by the lemma.

COROLLARY. *In case the absolute involution has double points the conic C^2 passes through them.*

THEOREM 2. *An orthogonal line reflection leaves the absolute involution invariant.*

Proof. If l is the axis of an orthogonal line reflection and L its center, let O be any point on l and P_1 any point not on l. The conic C^2 (cf. Lemma), which contains P_1, has O as center, and has the absolute involution as an involution of conjugate points, must have L and l as pole and polar. Hence, by the definition of pole and polar (§ 44, Vol. I) C^2 is transformed into itself by the harmonic homology having L and l as center and axis. Hence the absolute involution is transformed into itself by the orthogonal line reflection $\{Ll\}$.

THEOREM 3. *The product of two orthogonal line reflections whose axes are parallel is a translation parallel to any line perpendicular to the axes.*

Proof. Let the given line reflections be $\{L_1 l_1\}$ and $\{L_2 l_2\}$. Their axes meet in a point L' of l_∞, and L_1 and L_2 must be conjugate to L' with respect to the absolute involution. Hence $L_1 = L_2$. The product therefore leaves all points on l_∞ invariant and also all lines through L_1. Hence it is a translation parallel to any line through L_1.

THEOREM 4. *A translation,* T, *whose center is not a double point of the absolute involution, is a product of two orthogonal line reflections,* $\{Ll_2\}$, $\{Ll_1\}$, *where L is the center of the translation. If O is an arbitrary ordinary point and P the mid-point of the pair O and* T(O), *l_1 may be chosen as OL' and l_2 as PL', where L' is the conjugate of* L *with respect to the absolute involution. Or l_1 may be chosen as PL' and l_2 as the line joining* T(O) *to L'. A translation whose center is a double point of the absolute involution is a product of four orthogonal line reflections.*

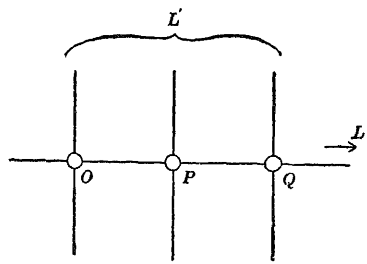


FIG. 46

Proof. If $l_1 = OL'$ and $l_2 = PL'$, the reflection $\{Ll_1\}$ leaves O invariant and $\{Ll_2\}$ carries O to T(O). Hence the translation $\{Ll_2\} \cdot \{Ll_1\}$ carries O to T(O), and, by Theorem 3, Chap. III, is identical with T.

If $l_1 = PL'$ and $l_2 = QL'$, where $Q =$ T(O), the reflection $\{Ll_1\}$ carries O to Q and $\{Ll_2\}$ leaves Q invariant. Hence, as before, $\{Ll_2\} \cdot \{Ll_1\} =$ T.

A translation whose center is a double point of the absolute involution can be expressed as a product of two translations with arbitrary points of l_∞ as centers (Theorem 8, Chap. III), and hence is expressible as a product of four orthogonal line reflections.

DEFINITION. If the axes of two orthogonal line reflections intersect in an ordinary point, O, the product is called a *rotation about* O, and the point O is called its *center*.

THEOREM 5. *A rotation which is the product of two orthogonal line*

Proof. Let the two line reflections be $\{L_1 l_1\}$ and $\{L_2 l_2\}$ and let O be the point of intersection of l_1 and l_2. Since l_1 and l_2 are orthogonal, L_1 is on l_2 and L_2 on l_1. The product $\{L_2 l_2\} \cdot \{L_1 l_1\}$ therefore leaves O and every point of l_∞ invariant. Moreover, it is of period two on the axis of either of the line reflections. Hence it is a homology of period two with O as center and l_∞ as axis, i.e. a point reflection.

DEFINITION. If a line l is perpendicular to a line m, the point of intersection of the two lines is called the *foot* of the perpendicular l. A line l is said to be the *perpendicular bisector* of a pair of points A and B if it is perpendicular to the line AB and its foot is the mid-point of the pair AB.

DEFINITION. A simple quadrangle $ABCD$ is said to be a *rectangle* if and only if the lines AB and CD are perpendicular to AD and BC.

EXERCISES

1. A parallelogram $ABCD$ is a rectangle if and only if the lines AB and AD are perpendicular.

2. The perpendicular bisectors of the point pairs AB, BC, CA of a triangle ABC meet in a point.

3. The perpendiculars from the vertices of a triangle to the opposite sides meet in a point.

4. The lines through the vertices of a triangle parallel to the transforms of the opposite sides by a fixed orthogonal line reflection are concurrent.

57. Displacements and symmetries. Congruence. DEFINITION. The product of an even number of orthogonal line reflections is called a *displacement*. The product of an odd number of orthogonal line reflections is called a *symmetry*.

THEOREM 6. *The set of all displacements form a self-conjugate subgroup of the parabolic metric group.*

Proof. That the displacements form a group is evident because (cf. § 26, Vol. I): (1) the identity is a displacement, being the product of any orthogonal line reflection by itself; (2) the inverse of a product of orthogonal line reflections is the product of the same set of line reflections taken in the reverse order; (3) the product of an even number of orthogonal line reflections by an even number of orthogonal line reflections is, by definition, a displacement.

The group of displacements is contained in the parabolic metric group by Theorem 2.

If $\{Ll\}$ is an orthogonal line reflection, Σ a similarity transformation, and $L' = \Sigma(L)$, $l' = \Sigma(l)$, then $\Sigma \cdot \{Ll\} \cdot \Sigma^{-1}$ is a harmonic homology with L' as center and l' as axis. But since L and the point at infinity of l are paired in the absolute involution, so are L' and the point at infinity of l'. Hence $\Sigma \cdot \{Ll\} \cdot \Sigma^{-1} = \{L'l'\}$ is an orthogonal line reflection.

If Λ_1 and Λ_2 are any two line reflections $\Sigma\Lambda_1\Lambda_2\Sigma^{-1} = \Sigma\Lambda_1\Sigma^{-1}\Sigma\Lambda_2\Sigma^{-1}$. A similar argument shows that $\Sigma\Lambda_1\Lambda_2 \cdots \Lambda_n \cdot \Sigma^{-1}$ is a product of n orthogonal line reflections whenever $\Lambda_1, \cdots, \Lambda_n$ are orthogonal line reflections and Σ is in the parabolic metric group. Hence the group of displacements is a self-conjugate subgroup of the parabolic metric group.

COROLLARY 1. *The set of all displacements and symmetries form a self-conjugate subgroup of the parabolic metric group.*

DEFINITION. Two figures such that one can be transformed into the other by a displacement are said to be *congruent*. Two figures such that one can be transformed into the other by a symmetry are said to be *symmetric.*

COROLLARY 2. *If a figure F_1 is congruent to a figure F_2, and F_2 to a figure F_3, then F_1 is congruent to F_3.*

COROLLARY 3. *If a figure F_1 is symmetric with a figure F_2, and F_2 is symmetric with a figure F_3, then F_1 is congruent to F_3.*

COROLLARY 4. *If a figure F_1 is symmetric with a figure F_2, and F_2 is congruent to a figure F_3, then F_1 is symmetric with F_3.*

Since translations and point reflections leave the absolute involution invariant, the definition of congruence given in this section includes the definitions in §§ 39 and 46 as special cases. Theorem 6 shows that the theory of congruence and symmetry in general belongs to the geometry of the parabolic metric group. It must be remembered, however, that the theory of congruence of point pairs on parallel lines belongs to the affine group. In other words, the part of the theory of congruence developed in Chap. III is independent of the choice of the absolute involution.

In case the absolute involution has double points, the theory of congruence of point pairs on the minimal lines (§ 56) is different from that on other lines. As will appear in the following sections the

theory on any line which is not minimal is essentially the same as that developed in Chap. III on the basis afforded by the group of translations and point reflections. On a minimal line, however, the set of points $[P]$ such that OP_0 is congruent to OP consists of all points on this line except the point O. For let I_1 and I_2 denote the double points of the absolute involution, I_1 being the one on the line OP_0. Let Q be a point of the line OI_2 distinct from O and from I_2, and let P be any point of OI_1 distinct from O and from I_1. If Λ_1 be the orthogonal line reflection whose center is the point at infinity of the line P_0Q and whose axis passes through O, and Λ_2 be the orthogonal line reflection whose center is the point at infinity of the line QP and whose axis passes through O, we have $\Lambda_1(P_0) = Q$ and $\Lambda_2(Q) = P$. Hence the rotation $\Lambda_2\Lambda_1$ transforms P_0 to P. Combining transformations of the form $\Lambda_2\Lambda_1$ with translations it is clear that we have

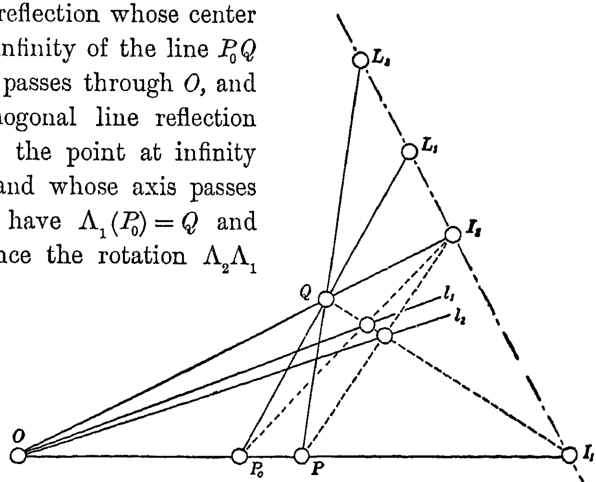


FIG. 47

THEOREM 7. *Any pair of points on a minimal line is congruent to any other pair of points on the same line.*

For example, if a mid-point of a pair AB were defined to be a point C such that AC is congruent to CB, we should have that whenever the line AB is minimal, the point C may be any point on this line different from A and B. The theorems on mid-points in Chap. III would in general have exceptional cases. It is to avoid this difficulty that we have adopted the definition of mid-point given in § 40, Chap. III. A similar remark applies to the definition of ratio of collinear point pairs in § 43, Chap. III.

DEFINITION. A parallelogram $ABCD$ whose sides do not pass through double points of the absolute involution and in which the point pair AB is congruent to the point pair AD is called a *rhombus*.

EXERCISES

1. Prove that the group of displacements and symmetries could be defined as the group of all collineations leaving invariant the set of all conics obtainable by translations from a fixed central conic.

2. The parabolic metric group consists of all projective collineations transforming the group of displacements into itself.

3. Two point pairs on nonminimal lines are symmetric if and only if they are congruent.

4. The perpendicular bisector of a point pair AB contains all points P such that AP is congruent to BP.

5. The simple quadrangle $ABCD$ is a rhombus if and only if the lines AC and BD are the perpendicular bisectors of the point pairs BD and AC respectively.

6. A parallelogram $ABCD$ is a rectangle if and only if the point pair AC is congruent to the point pair BD.

7. Specialize the quadrangle-quadrilateral configuration (§ 18, Vol. I) to the case where the vertices of the quadrangle are the vertices of a square.

58. Pairs of orthogonal line reflections. THEOREM 8. *If Λ_1, Λ_2, Λ_3 are three orthogonal line reflections whose axes pass through a point O (ordinary or ideal), the product $\Lambda_3\Lambda_2\Lambda_1$ is an orthogonal line reflection whose axis passes through O.*

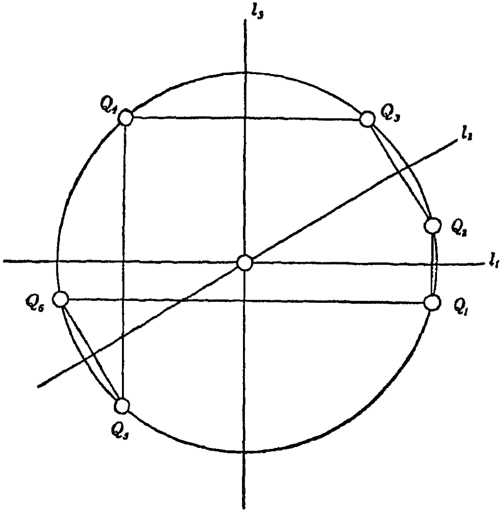


FIG. 48

Proof. In case the three axes are parallel, the product $\Lambda_3\Lambda_2$ is a translation, and so by Theorem 4 is expressible in the form $\Lambda_4\Lambda_1$, where Λ_4 is an orthogonal line reflection whose axis is parallel to the other axes. Hence

$$\Lambda_3\Lambda_2\Lambda_1 = \Lambda_4\Lambda_1\Lambda_1 = \Lambda_4.$$

In case two of the axes are not parallel, the third axis must pass through their common point O. Let P be any point not collinear with O and a circular point. Let C^2 be the conic, existent and unique according to the lemma of § 56, which passes through P, has O as

points. If Q_1 be any point of C^2, let $\Lambda_1(Q_1) = Q_2$, $\Lambda_2(Q_2) = Q_3$, $\Lambda_3(Q_3) = Q_4$, $\Lambda_1(Q_4) = Q_5$, $\Lambda_2(Q_5) = Q_6$.

According to this construction the line Q_1Q_2 is parallel to Q_4Q_5 and Q_2Q_3 to Q_5Q_6, where in case $Q_i = Q_j$, the line Q_iQ_j is taken to mean the tangent to C^2 at Q_i. Hence, by Pascal's theorem (Chap. V, Vol. I) or one of its degenerate cases, it follows that Q_3Q_4 is parallel to Q_6Q_1. Hence

$$\Lambda_3(Q_6) = Q_1$$

and

$$(\Lambda_3\Lambda_2\Lambda_1)^2(Q_1) = Q_1.$$

Since Q_1 is an arbitrary point of C^2,

$$(\Lambda_3\Lambda_2\Lambda_1)^2 = 1.$$

The transformation $\Lambda_3\Lambda_2\Lambda_1$ is not the identity, because it cannot leave invariant a point, different from O, of the axis of Λ_1 unless $\Lambda_2 = \Lambda_3$, and in the latter case the product is equal to Λ_1. Since $\Lambda_3\Lambda_2\Lambda_1$ leaves invariant the line Q_1Q_4 (or the tangent at Q_1, if $Q_1 = Q_4$), it leaves invariant the point at infinity of this line and also the line through O perpendicular to it. As $\Lambda_3\Lambda_2\Lambda_1$ is of period two, it follows that it is an orthogonal line reflection.

COROLLARY 1. *If Λ_1, Λ_2, and Λ_3 are any three orthogonal line reflections whose axes meet in a point or are parallel, there exists an orthogonal line reflection Λ_4 such that $\Lambda_2\Lambda_1 = \Lambda_3\Lambda_4$, and an orthogonal line reflection Λ_5 such that $\Lambda_2\Lambda_1 = \Lambda_5\Lambda_3$.*

Proof. By the theorem, Λ_4 exists such that

$$\Lambda_3\Lambda_2\Lambda_1 = \Lambda_4.$$

Hence

$$\Lambda_2\Lambda_1 = \Lambda_3\Lambda_4.$$

In like manner, Λ_5 exists such that

$$\Lambda_2\Lambda_1\Lambda_3 = \Lambda_5.$$

Hence

$$\Lambda_2\Lambda_1 = \Lambda_5\Lambda_3.$$

COROLLARY 2. *The product of any odd number of orthogonal line reflections whose axes meet in a point or are parallel is an orthogonal line reflection.*

Proof. By the theorem, whenever $n \geqq 3$, the product of n orthogonal line reflections whose axes are concurrent reduces to a product of $n - 2$. Thus, if n is odd, the number of line reflections can be

the product to two. Thus we have

COROLLARY 3. *The product of any even number of orthogonal line reflections is a rotation in case their axes meet in a point, and is a translation in case the axes are parallel.*

COROLLARY 4. *An orthogonal line reflection is not a displacement.*

COROLLARY 5. *The set of all rotations having a common center is a commutative group.*

Proof. A rotation is defined as a product of two orthogonal line reflections whose axes meet in an ordinary point. So, by definition, the identity is a rotation, and the inverse of a rotation $\Lambda_2\Lambda_1$ is the rotation $\Lambda_1\Lambda_2$. The product of two rotations is a rotation by Cor. 3. Hence the rotations having a given point as center form a group. To show that any two of these rotations are commutative amounts to showing that

$$(1) \qquad \Lambda_4\Lambda_3\Lambda_2\Lambda_1 = \Lambda_2\Lambda_1\Lambda_4\Lambda_3$$

whenever the Λ's are orthogonal line reflections whose axes concur. By the theorem we have

$$\Lambda_4\Lambda_3\Lambda_2 = \Lambda_2\Lambda_3\Lambda_4,$$

and hence

$$(2) \qquad \Lambda_4\Lambda_3\Lambda_2\Lambda_1 = \Lambda_2\Lambda_3\Lambda_4\Lambda_1.$$

But since

$$\Lambda_3\Lambda_4\Lambda_1 = \Lambda_1\Lambda_4\Lambda_3,$$

$$\Lambda_2\Lambda_3\Lambda_4\Lambda_1 = \Lambda_2\Lambda_1\Lambda_4\Lambda_3,$$

which combined with (2) gives (1).

THEOREM 9. *Any displacement leaving a point O invariant is a rotation about O.*

Proof. The given displacement is a product of an even number, n, of orthogonal line reflections, $\Lambda_n \cdots \Lambda_1$. Let Λ_i' be the line reflection whose axis is the line through O parallel to the axis of Λ_i. Then the product $T_i = \Lambda_i\Lambda_i'$ is a translation (Theorem 3) and

$$\Lambda_i = T_i\Lambda_i'.$$

Thus

$$\Lambda_n \cdots \Lambda_1 = T_n\Lambda_n' \cdots T_1\Lambda_1',$$

where each T_i is a translation. But by Cor. 2, Theorem 11, Chap. III, if Σ is any affine collineation, $T_i\Sigma = \Sigma T_i'$, where T_i' is a translation or the identity. Hence

$$\Lambda_n \cdots \Lambda_1 = \Lambda_n' \cdots \Lambda_1' T_n' \cdots T_1'.$$

But since $\Lambda_n \cdots \Lambda_1$ and $\Lambda_n' \cdots \Lambda_1'$ leave O invariant, the product $\mathrm{T}_n' \cdots \mathrm{T}_1'$ leaves O invariant, and hence, by Theorem 3, Chap. III, is the identity. Hence

$$\Lambda_n \cdots \Lambda_1 = \Lambda_n' \cdots \Lambda_1',$$

where $\Lambda_1', \cdots, \Lambda_n'$ are orthogonal line reflections whose axes pass through O. By Cor. 3, Theorem 8, $\Lambda_n' \cdots \Lambda_1'$ is a rotation about O.

59. The group of displacements. THEOREM 10. *Let O be an arbitrary point. Any displacement can be expressed in the form* PT, *where* P *is a rotation about O and* T *a translation.*

Proof. By precisely the argument used in the last theorem the given displacement can be expressed in the form

$$\Lambda_{2n}' \cdots \Lambda_1' \mathrm{T}_{2n}' \cdots \mathrm{T}_1',$$

where $\Lambda_i' (i = 1, \cdots, 2n)$ is an orthogonal line reflection whose axis passes through O, and $\mathrm{T}_i' (i = 1, \cdots, 2n)$ is a translation or the identity. The product $\mathrm{T}_{2n}' \cdots \mathrm{T}_1'$ is, by Theorem 6, Chap. III, a translation. By Cor. 3, Theorem 8, $\Lambda_{2n}' \cdots \Lambda_1'$ is a rotation or a translation. Since it leaves O invariant, it is a rotation.

COROLLARY 1. *Any displacement can also be expressed in the form* T′P′, *where* T′ *is a translation and* P′ *a rotation with O as center.*

COROLLARY 2. *Any symmetry is a product of a line reflection whose axis contains an arbitrary point and a translation.*

THEOREM 11. *Any displacement, except a translation having a double point of the absolute involution as center, is a product of two orthogonal line reflections.*

Proof. Let O be an arbitrary point. By the last theorem the given displacement reduces to PT, where T is a translation and P a rotation about O. If the center, L, of T is not a double point of the absolute involution, by Theorem 4,

$$\mathrm{T} = \{Ll_2\} \cdot \{Ll_1\},$$

where l_1 and l_2 meet l_∞ in the conjugate of L relative to the absolute involution and where l_2 passes through O. By Cor. 1, Theorem 8, there exists an orthogonal line reflection $\{Mm\}$ such that

$$\mathrm{P} = \{Mm\} \cdot \{Ll_2\}.$$

Hence

$$\begin{aligned}\mathrm{PT} &= \{Mm\} \cdot \{Ll_2\} \cdot \{Ll_2\} \cdot \{Ll_1\} \\ &= \{Mm\} \cdot \{Ll_1\}.\end{aligned}$$

If P is not the identity, it is clear that m and l_1 cannot be parallel, and hence PT is a rotation.

In case T is a translation whose center is a double point of the absolute involution, it can be expressed (Theorem 8, Chap. III) as a product of two translations T_1, T_2 whose centers are not double points of the absolute involution. Hence, if P is not the identity, PT_2 is a rotation, and thus PT_2T_1 is also a rotation. In case P is the identity, we have the exceptional case noted in the theorem.

COROLLARY. *A displacement is either a rotation or a translation.*

The following two theorems have the same relation to the parabolic metric group and the group of displacements, respectively, that the fundamental theorem of projective geometry (Assumption P) has to the projective group on a line.

THEOREM 12. *A transformation of the parabolic metric group leaving invariant two ordinary points not collinear with a double point of the absolute involution is either an orthogonal line reflection or the identity.*

Proof. Denote the given fixed points by O and P, and let C^2 be the conic through P having O as center and the absolute involution as an involution of conjugate points. Since C^2 is uniquely determined by these conditions (cf. the lemma in § 56), it is left invariant by the given transformation Γ. Now Γ leaves O, P, and the point at infinity of the line OP invariant. Hence the line OP is point-wise invariant, and every line l perpendicular to it is transformed into itself. Since C^2 is also invariant and each of the lines perpendicular to OP meets C^2 in at most two points, Γ is either the identity or of period two. If of period two, it is evidently an orthogonal line reflection.

THEOREM 13. *A displacement leaving invariant a point O and a line l containing O but not containing a double point of the absolute involution is either the identity or a point reflection with O as center.*

Proof. Let P be any ordinary point of l distinct from O, and let C^2 be the conic through P having O as center and the absolute involution as an involution of conjugate points. A displacement leaving O invariant, being a product of two orthogonal line reflections whose axes meet in O, must leave C^2 invariant. Hence it either leaves P invariant or transforms it into the other point in which the line OP meets C^2. In the first case the transformation must, by Theorem 12

and Cor. 4, Theorem 8, reduce to the identity. In the second case the given displacement, which we shall denote by Δ, multiplied by the orthogonal line reflection Λ whose axis is the line through O perpendicular to OP, leaves P invariant. Hence, by Theorem 12,

$$\Delta\Lambda = \Lambda',$$

where Λ' is a line reflection having OP as axis or the identity. Hence

$$\Delta = \Lambda'\Lambda.$$

Since Δ cannot be a line reflection, Λ' cannot be the identity. Since the axes of Λ and Λ' are perpendicular, Δ is a point reflection.

EXERCISES

1. A displacement which carries a point A to a point B and has a point O (ordinary or not) as center is, if the line OA is not minimal, the product of an orthogonal line reflection whose axis is OA followed by one whose axis is the line joining O to the mid-point of the pair AB.

2. If three of the perpendicular bisectors of the point pairs AB, BC, CD, DA of a simple quadrangle meet in a point, the fourth perpendicular bisector passes through this point.

*3. Any affine transformation which leaves a central conic invariant is a line reflection whose center and axis are pole and polar with regard to the conic or a product of two such line reflections.

*4. In case the absolute involution is without double points, the group of displacements can be defined as the group of transformations common to the parabolic metric group and the equiaffine group. Thus two ordered point triads are congruent if they are both equivalent and similar. Develop the theory of congruence on this basis, and show what difficulties arise in case the absolute involution has double points.

60. Circles. DEFINITION. A *circle* is the set of all points $[P]$ such that the point pairs OP, where O is a fixed point, are all congruent to a fixed point pair OP_0, provided that the line OP_0 does not contain a double point of the absolute involution. The point O is called the *center* of the circle.

Since the displacements form a group, it is clear that P_0 may be any one of the points P. It has already been proved (§ 57) that if the line OP_0 contained an invariant point of the absolute involution, the set $[P]$ would consist of all ordinary points, except O, of the line OP_0.

THEOREM 14. *A circle consists of the ordinary points of a conic section having the pairs of the absolute involution as pairs of conjugate points. The center of the circle is the pole of l_∞ with respect to the circle.*

Proof. Let O be the center of the circle and P_0 any point of the circle. The circle consists of all points obtainable from P_0 by displacements which leave O invariant. If one of the line reflections of which each of these displacements is a product be taken to have OP_0 as axis (Cor. 1, Theorem 8), it follows that the circle consists of the points obtainable from P_0 by orthogonal line reflections whose axes pass through O. But the system of points so obtained is identical by construction with the ordinary points of the conic referred to in the lemma of § 56.

COROLLARY. *In case the absolute involution has no double points, every circle is a conic section. In case the circular points exist, they and the points of any circle form a conic section.*

THEOREM 15. *The ordinary points of any proper conic, with regard to which the pairs of the absolute involution are pairs of conjugate points, form a circle.*

Proof. A conic C^2 with regard to which the pairs of the absolute involution are conjugate points cannot be a parabola, since all points of l_∞ are conjugate to the point of contact of a parabola. Hence C^2 has an ordinary point O as center. Let P be any point of C^2. By definition there is one and only one circle through P which has O as a center. By Theorem 14, this circle is a conic through P having O as center and the pairs of the absolute involution as pairs of conjugate points. By the lemma of § 56 there is only one such conic. Hence the circle through P with O as center contains the ordinary points of C^2.

THEOREM 16. *Three noncollinear points, no two of which are on a minimal line, are contained in one and only one circle.*

Proof. Let the three points be P_0, P_1, and P_2. Let L_∞ be the point at infinity of the line P_0P_1 and l the perpendicular bisector of the point pair P_0P_1. The polar of L_∞ with regard to any circle through P_0 and P_1 must, by Theorem 14, pass through the mid-point of P_0P_1 and the conjugate of L_∞ in the absolute involution. Hence the polar of L_∞ with regard to any circle through P_0 and P_1 must be l. In like manner, the polar of the point at infinity M_∞ of the line P_1P_2 with regard to any circle containing P_1 and P_2 must be the perpendicular bisector m of P_1P_2. Since the points P_0, P_1, P_2 are not collinear, l and

$L_\infty M_\infty = l_\infty$ with regard to any circle through P_0, P_1, and P_2. Since, by definition, there is one and only one circle through P with O as center, there cannot be more than one circle through P_0, P_1, and P_2.

Since the product of the orthogonal line reflection with OP_0 as axis by that with l as axis transforms the point pair OP_0 into the point pair OP_1, the circle through P_0 with O as center contains P_1. A like argument shows that it contains P_2. Hence there is one circle containing P_0, P_1, and P_2.

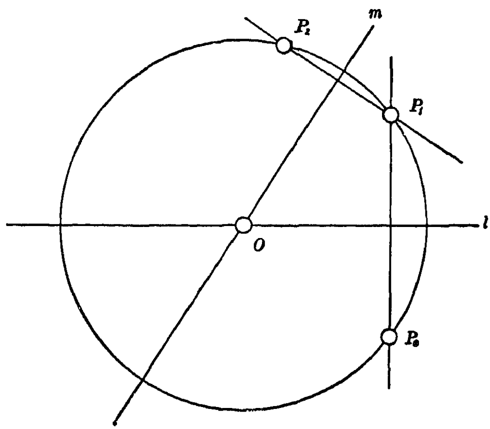


FIG. 49

Observe that we do not prove at this stage that a circle has a point on every line through its center. This could not be done without further hypotheses on the nature of the plane than we are making at present.

EXERCISES

1. The locus of the points of intersection of the lines through a point A with the perpendicular lines through a point B, not on a minimal line through A, is a circle whose center is the mid-point of the pair AB.

2. A tangent to a circle is perpendicular to the diameter through the point of contact.

3. Any two conjugate diameters of a circle are orthogonal.

4. If the tangents at two points A and B of a circle meet in a point O, the pairs OA and OB are congruent.

5. If l is the perpendicular bisector of a point pair AB, then the circles through A and B meet l in pairs of an involution whose center (§ 43) is the mid-point of AB.

6. The system of all circles having a common center meet any line in the pairs of an involution.

7. A parallelogram which circumscribes a circle must be a rhombus.

8. A parallelogram inscribed in a circle is a rectangle.

9. If two circles have two points in common, the pair of tangents at one common point is symmetric to the pair of tangents at the other.

10. The feet of the perpendiculars from any point of a circle to the sides of an inscribed triangle are collinear.

If the coördinate system be chosen so that (0, 1, 0) and (0, 0, 1) are conjugate points in this involution, the bilinear equation reduces to

$$ax_1\bar{x}_1 + cx_2\bar{x}_2 = 0. \tag{3}$$

Here the point (0, 1, 1) is paired with the point $(0, c, -a)$. In case the involution contains two pairs of points which are harmonically conjugate, one pair may be chosen as (0, 1, 0) and (0, 0, 1) and the other pair as (0, 1, 1) and (0, 1, −1). In that case (3) reduces to

$$x_1\bar{x}_1 + x_2\bar{x}_2 = 0. \tag{4}$$

For the rest of this section we assume that the absolute involution contains two pairs of points which are harmonically conjugate with respect to each other. Such involutions exist in every plane satisfying Assumption H_0, since any two distinct collinear pairs of points determine an involution. Hence this assumption is no restriction on the nature of the plane in which we are working. It is, moreover, easy to replace the formulas which we shall obtain from (4) by the more general but more cumbersome formulas based on (3).

The equations of the transformation required to change (3) into (4) are

$$x_0' = x_0, \quad x_1' = \sqrt{c}\, x_1, \quad x_2' = \sqrt{a}\, x_2.$$

Hence it is clear that in the complex geometry (§ 5) every involution may be reduced to the form (4), and in the real geometry only those involutions can be reduced to this form which are such that $a/c > 0$. The involutions of the latter type are direct (§ 18).

The equations of the affine group are

$$\begin{aligned} x_0' &= x_0, \\ x_1' &= c_1x_0 + a_1x_1 + b_1x_2, \\ x_2' &= c_2x_0 + a_2x_1 + b_2x_2, \end{aligned} \tag{5}$$

and if the involution (4) is to be transformed into itself, all pairs x_1, x_2 and $\bar{x}_1$, $\bar{x}_2$ which satisfy

$$x_1\bar{x}_1 + x_2\bar{x}_2 = 0$$

must also satisfy

$$(a_1x_1 + b_1x_2)(a_1\bar{x}_1 + b_1\bar{x}_2) + (a_2x_1 + b_2x_2)(a_2\bar{x}_1 + b_2\bar{x}_2) = 0,$$

which is the same as

$$(a_1^2 + a_2^2)x_1\bar{x}_1 + (a_1b_1 + a_2b_2)(x_1\bar{x}_2 + x_2\bar{x}_1) + (b_1^2 + b_2^2)x_2\bar{x}_2 = 0.$$

$$\begin{cases} a_1^2 + a_2^2 = b_1^2 + b_2^2 \neq 0,\end{cases}$$

are the necessary and sufficient conditions that (5) leave (4) invariant. Combining these two equations, we obtain

$$a_1^2 a_2^2 + a_2^4 - b_1^2 a_2^2 - a_1^2 b_1^2 = 0$$

or

$$(a_1^2 + a_2^2)(a_2^2 - b_1^2) = 0.$$

Thus we infer $a_2 = \pm b_1$ and $a_1 = \mp b_2$. Hence

THEOREM 19. *The equations of the parabolic metric group are*

$$(6) \qquad \begin{aligned} x' &= \alpha x + \beta y + \gamma_1, \\ y' &= \epsilon(-\beta x + \alpha y) + \gamma_2, \end{aligned}$$

where $\epsilon^2 = 1$.

Any conic section has an equation of the form (§ 66, Vol. I)

$$(7) \quad a_{00}x_0^2 + a_{11}x_1^2 + a_{22}x_2^2 + 2\,a_{01}x_0x_1 + 2\,a_{02}x_0x_2 + 2\,a_{12}x_1x_2 = 0,$$

which determines on the line $x_0 = 0$ an involution whose double elements satisfy

$$a_{11}x_1^2 + a_{22}x_2^2 + 2\,a_{12}x_1x_2 = 0.$$

Comparing with (4), we have that a circle must satisfy the condition

$$a_{11} = a_{22} \neq 0, \qquad a_{12} = 0.$$

If this circle is to have (1, 0, 0) as center, i.e. as pole of $x_0 = 0$, the equation (7) must also satisfy the condition

$$a_{01} = 0 = a_{02}.$$

Thus, returning to nonhomogeneous coördinates, the equation of a circle with the origin as center must be of the form*

$$(8) \qquad x^2 + y^2 = k.$$

According to § 59, the transformations of the parabolic metric group leaving such a circle invariant are all displacements or symmetries, and, moreover, all displacements and symmetries leaving the origin invariant leave this circle invariant. Substituting (6) in (8), we see that a displacement or symmetry leaving the origin invariant is of the form

$$\begin{aligned} x' &= \alpha x + \beta y, \\ y' &= \epsilon(-\beta x + \alpha y), \end{aligned} \qquad \alpha^2 + \beta^2 = 1.$$

* This argument does not prove that every equation of this form represents a

Since any displacement or symmetry is expressible as the resultant of one leaving the origin invariant and a translation (Theorem 10, Cor. 1), we have

THEOREM 20. *The equations of the group of displacements and symmetries are*

$$(9) \qquad \begin{aligned} x' &= \alpha x + \beta y + \gamma_1, \\ y' &= \epsilon(-\beta x + \alpha y) + \gamma_2, \end{aligned}$$

where $\alpha^2 + \beta^2 = 1$ *and* $\epsilon^2 = 1$.

By § 54, Vol. I, a transformation of the form (9) effects an involution on l_∞ if and only if $\epsilon = -1$. By Theorem 10, Cor. 2, any symmetry leaving the origin invariant is a line reflection. Hence

THEOREM 21. *The displacements are the transformations of the type* (9) *for which* $\epsilon = 1$ *and the symmetries those for which* $\epsilon = -1$.

EXERCISES

1. The equation of a circle containing the point (a_2b_2) and having the point (a_1b_1) as center is

$$(x - a_1)^2 + (y - b_1)^2 = (a_2 - a_1)^2 + (b_2 - b_1)^2.$$

2. Two lines $ax + by + c = 0$ and $a'x + b'y + c' = 0$ are orthogonal if and only if $aa' + bb' = 0$.

3. In case the absolute involution has double points, the equiaffine transformations of the parabolic metric group are of the form (9), where $\alpha^2 + \beta^2 = \epsilon$ and $\epsilon = \pm 1$.

63. Introduction of order relations. Let us now assume that the plane which we are considering is an ordered plane in the sense of § 15. We may therefore apply the results of Chap. II, particularly of §§ 28–30. Let us also assume that the absolute involution satisfies the condition referred to in § 62, that there exist two pairs of points conjugate with regard to the absolute involution which separate each other harmonically. By Theorem 9, Chap. II, and its corollaries, it follows that any two pairs of the absolute involution separate each other, and that the absolute involution has no double points.* This result may conveniently be put in the following form:

THEOREM 22. *Two pairs of perpendicular lines intersecting in the same point separate each other. No line is perpendicular to itself.*

* The geometry arising from the hyperbolic case has been studied by Wilson and Lewis in the article referred to in § 42.

The restrictions which we have just introduced enable us to state the fundamental theorem (Theorem 13) about the group of displacements in the following more precise form:

THEOREM 23. *The only displacement leaving a ray invariant is the identity.*

Proof. Let A be the origin and B any point of the ray. Since any collineation preserves order relations, A is transformed into itself. Since the line AB is invariant, the displacement is a point reflection or the identity (Theorem 13). But a point reflection would change B into a point of the ray opposite to the ray AB, and thus not leave the ray AB invariant.

With the aid of this theorem we can complete the set of fundamental theorems on congruent triangles, the first two of which were given in § 61.

THEOREM 24. *Two triangles ABC and $A'B'C'$ are congruent if the point pairs AB, AC and the angle $\measuredangle CAB$ are congruent respectively to the point pairs $A'B'$, $A'C'$ and the angle $\measuredangle C'A'B'$.*

Proof. Since the angle* $\measuredangle CAB$ is congruent to the angle $\measuredangle C'A'B'$, there exists a displacement Δ_1 carrying A to A' and the rays AC and AB to $A'C'$ and $A'B'$ respectively. Since the point pair AB is congruent to $A'B'$, there is also a displacement Δ_2 carrying A to A' and B to B', and since AC is congruent to $A'C'$, there is a displacement Δ_3 carrying A to A' and C to C'. By Theorem 23, $\Delta_1 = \Delta_2$ and $\Delta_1 = \Delta_3$. Hence the displacement Δ_1 carries the triangle ABC to $A'B'C'$.

EXERCISES

1. Two triangles ABC and $A'B'C'$ are congruent if the point pair AB is congruent to the point pair $A'B'$ and the angles $\measuredangle CAB$ and $\measuredangle CBA$ are congruent respectively to the angles $\measuredangle C'A'B'$ and $\measuredangle C'B'A'$.

2. If two triangles ABC and $A'B'C'$ are congruent in such a way that A corresponds to A' and B to B', the angles $\measuredangle ABC$, $\measuredangle BCA$, $\measuredangle CAB$ are congruent to the angles $\measuredangle A'B'C'$, $\measuredangle B'C'A'$, $\measuredangle C'A'B'$ respectively.

3. If two triangles ABC and $A'B'C'$ are symmetric in such a way that A corresponds to A' and B to B', the angles $\measuredangle ABC$, $\measuredangle BCA$, $\measuredangle CAB$ are congruent to the angles $\measuredangle C'B'A'$, $\measuredangle A'C'B'$, $\measuredangle B'A'C'$ respectively.

4. Let A, B, C be three collinear points and P_∞ the point at infinity of the line joining them; B is between A and C if and only if

$$0 < \text{R}(P_\infty A, CB) < 1.$$

5. An orthogonal line reflection interchanges the two sides of its axis.

* Note that an angle is an *ordered* pair of rays (§ 28).

64. The real plane. Let us finally assume that we are dealing with the geometry of reals. In consequence, we have the theorem (§ 4) that any one-dimensional projectivity which alters sense (i.e. for which $\Delta < 0$) has two double elements. This may be put into the following form as a theorem of the affine geometry.

THEOREM 25. *If A_1 and A_2 are any two points of an ellipse, any line l, meeting the line A_1A_2 in a point between A_1 and A_2, meets the ellipse in two points.*

*Proof.** Let us denote the given ellipse by E^2, and let A be a variable point on it. Let L_1 and L_2 be the points in which l is met by A_1A and A_2A respectively, and let Q_1 and Q_2 be the points in which l_∞ is met by A_1A and A_2A respectively. Also let Q_3 be the point in which A_1L_2 meets l_∞. By construction, and by the definition of a conic,

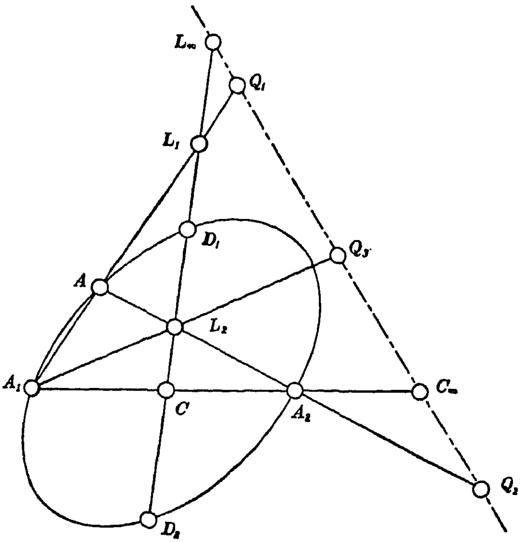


FIG. 51

$$(10) \qquad [L_1] \overset{A_1}{\underset{\wedge}{=}} [Q_1] \underset{\wedge}{=} A_1[A] \barwedge A_2[A] \underset{\wedge}{=} [Q_2] \overset{A_2}{\underset{\wedge}{=}} [L_2] \overset{A_1}{\underset{\wedge}{=}} [Q_3].$$

The projectivity $[Q_1] \barwedge [Q_2]$ is direct, because, by the remark at the beginning of this section, if the projectivity altered sense it would have two double points, and these, by the definition of the projectivity, would be points of intersection of l_∞ with E^2, contrary to the hypothesis that E^2 is an ellipse.

Let C and C_∞ be the points of intersection of A_1A_2 with l and l_∞ respectively. Also let L_∞ be the point at infinity of l. Then, by the hypothesis that C is between A_1 and A_2,

But, by construction, $A_1CA_2C_\infty \overset{L_2}{\underset{\wedge}{=}} Q_3L_\infty Q_2C_\infty$.

Hence, by Theorem 6, Chap. II,

$$S(C_\infty L_\infty Q_2) \neq S(C_\infty L_\infty Q_3).$$

But the points C_∞, L_∞, Q_2 are carried to C_∞, L_∞, Q_3, respectively, by the projectivity $[Q_2]\overline{\wedge}[Q_3]$, indicated in (10). Hence the projectivity $[Q_2]\overline{\wedge}[Q_3]$ is opposite. Since $[Q_1]\overline{\wedge}[Q_2]$ is direct, $[Q_1]\overline{\wedge}[Q_3]$ is opposite. From this, since Q_1 and Q_3 are carried by a perspectivity with A_1 as center to L_1 and L_2 respectively, it follows (Theorem 6, Chap. II) that the projectivity

$$[L_1]\overline{\wedge}[L_2]$$

is opposite. By the remark at the beginning of the section this projectivity must therefore have two double points, and by the definition of the projectivity these double points must be points of intersection of l with E^2.

COROLLARY 1. *The points in which l meets the ellipse are separated by A_1 and A_2 relative to the order relations on the ellipse.*

Proof. Let D_1 and D_2 (fig. 51) be the two points in which l meets the ellipse, and let A, A_1, A_2, etc. have the meanings given them in the proof of the theorem. Then since the projectivity $[L_1]\overline{\wedge}[L_2]$ is opposite,

$$S(D_1D_2L_1) \neq S(D_1D_2L_2).$$

Hence the lines AD_1 and AD_2 separate the lines AA_1 and AA_2, which, according to the definition in § 20, implies that the pair of points D_1D_2 separates the pair A_1A_2 on the ellipse.

COROLLARY 2. *The points in which l meets the ellipse are on opposite sides of the line A_1A_2.*

Proof. Let a be the tangent at A_1. By the first corollary the lines a and A_1A_2 separate the lines A_1D_1 and A_1D_2. Hence, if A' denote the point in which a meets D_1D_2, D_1 and D_2 separate A' and C. Now A' is not between D_1 and D_2, because if it were, the line a would meet the ellipse in two points instead of only in one. Hence C is between D_1 and D_2, and hence D_1 and D_2 are on opposite sides of l.

THEOREM 26. *A rotation which transforms a given circle into itself*

Proof. Let the given triad of points be A, B, C, let O be any other point of the circle, and let $A_\infty, B_\infty, C_\infty$ be the points at infinity of the lines OA, OB, OC respectively; let $O', A', B', C', A'_\infty, B'_\infty, C'_\infty$ be the points to which $O, A, B, C, A_\infty, B_\infty, C_\infty$, respectively, are carried by the given rotation; let $A''_\infty, B''_\infty, C''_\infty$ be the points at infinity of the lines OA', OB', OC' respectively.

The given rotation effects on l_∞ a transformation which is the product of two hyperbolic involutions. Hence $S(A_\infty B_\infty C_\infty) = S(A'_\infty B'_\infty C'_\infty)$. As in the proof of Theorem 25, the projectivity $A'_\infty B'_\infty C'_\infty \barwedge A''_\infty B''_\infty C''_\infty$ is direct because otherwise it would have double points and these would be common to the circle and l_∞. Hence $S(A'_\infty B'_\infty C'_\infty) = S(A''_\infty B''_\infty C''_\infty)$ and, therefore, $S(A_\infty B_\infty C_\infty) = S(A''_\infty B''_\infty C''_\infty)$. Projecting from O, we have, by the definition of sense on a conic (§ 20), that

$$S(ABC) = S(A'B'C').$$

Theorem 26, which is here proved only for a real space, can be proved for any ordered space by the methods of the next chapter. This theorem states one of the most intuitionally immediate properties of a rotation. In fact, most of the older discussions of the notions of sense describe sense, without further explanation, as "sense of rotation."

EXERCISES

1. If $\measuredangle AOB$ is any angle, and PQ any ray, there is one and only one ray PR on a given side of the line PQ such that $\measuredangle AOB$ is congruent or symmetric to $\measuredangle QPR$.

***2.** Prove that Theorem 25 is not true in a space satisfying Assumptions A, E, H, Q.

65. Intersectional properties of circles. THEOREM 27. *If A and B are any two distinct points, then on any ray having a point O as origin there is one and only one point P such that the pair AB is congruent to the pair OP.*

Proof. Let B_1 be the point to which B is carried by the translation which carries A to O. The circle through B_1 with O as center contains all points Q such that OQ is congruent to AB. Let B_2 be the point to which B_1 is transformed by a point reflection with O as center. Then since O is between B_1 and B_2, any line l through O (and distinct from OB_1) must meet the circle in two points, according to Theorem 25. But by Theorem 23 neither of the rays on l which have O as origin

rays contains just one point of the circle. Hence each ray with O as origin contains a single point P such that AB is congruent to OP.

Combining this theorem with Theorem 23, we have

THEOREM 28. *There is one and only one displacement carrying a given ray to a given ray.*

This result characterizes the group of displacements in the same way that the proposition that there is a unique projectivity of a one-dimensional form carrying any ordered triad of elements to any ordered triad characterizes the one-dimensional projective group.

THEOREM 29. *If two circles are such that the line joining their centers meets them in two point pairs which separate each other, the circles have two points in common, one on each side of the line joining the centers.*

Proof. Let the two circles be C_1^2 and C_2^2, and let them meet the line joining the centers in the pairs P_1Q_1 and P_2Q_2 respectively. Let A be the center (§ 43) of the involution Γ in which P_1Q_1 and P_2Q_2 are pairs, and let a be the perpendicular to the line P_1P_2 at A.

Since P_1 and Q_1 separate P_2 and Q_2, the ordered triads $P_1Q_1P_2$ and $Q_1P_1Q_2$ are in the same sense. The involution Γ interchanges these two triads and hence transforms any triad into a triad in the same sense. Hence A is between P_1 and Q_1. Hence, by Theorem 25, the line meets the circle C_1^2 in two points A_1 and A_2; and by the second corollary of this theorem, A_1 and A_2 are on opposite sides of the line P_1Q_1.

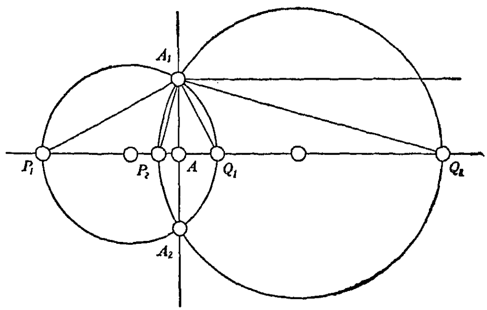


FIG. 52

The lines A_1P_1 and A_1Q_1 are orthogonal since P_1 and Q_1 are the ends of the diameter of a circle through A_1. The line A_1A is orthogonal to the line through A_1 parallel to P_1Q_1. Hence the involution Γ is perspective with the involution of pairs of orthogonal lines through A_1. Hence A_1, P_2, and Q_2 are on a circle whose center is on the line P_1Q_1. By Theorem 16 this circle must be C_2^2. Hence C_1^2 and C_2^2 have A_1 in common. A similar argument shows that A_2 is on C_1^2 and C_2^2.

the well-known equations for the "rigid motions" of elementary Euclidean geometry. Hence *the geometry of the parabolic metric group in a real plane is the Euclidean geometry.*

This result can also be established by considering a set of postulates from which the theorems of Euclidean geometry are deducible and proving that these postulates are theorems of the parabolic metric geometry. It then follows that all the theorems of Euclidean geometry are true in the parabolic metric geometry.

As a set of assumptions for Euclidean geometry of three dimensions we may choose the ordinal assumptions I–IX which are stated in § 29, together with the assumptions of congruence (X–XVI) stated below. For our immediate purpose, however, a set of assumptions for Euclidean plane geometry is needed. To obtain such a set we merely replace VII and VIII by the following:

$\overline{\text{VII}}$. *All points are in the same plane.*

Thus our set of postulates for Euclidean plane geometry is I–VI, $\overline{\text{VII}}$, IX–XVI.

Assumptions X–XVI make use of a new undefined relation between ordered point pairs which is indicated by saying "AB is congruent to CD." It must be verified that the new assumptions are valid when this relation is identified with the relation of congruence defined above.

X. *If $A \neq B$, then on any ray whose origin is a point C there is one and only one point D such that AB is congruent to CD.*

Proof. This is the same as Theorem 27.

XI. *If AB is congruent to CD and CD is congruent to EF, then AB is congruent to EF.*

Proof. This is a consequence of the fact that the displacements form a group.

XII. *If AB is congruent to $A'B'$, and BC is congruent to $B'C'$ and $\{ABC\}$ and $\{A'B'C'\}$, then AC is congruent to $A'C'$.*

Proof. By Theorem 28, there is a unique displacement which carries A and B to A' and B' respectively. This displacement carries

order relations. Moreover, the point C' so obtained is such that BC is congruent to $B'C'$ and AC to $A'C'$; and, by Theorem 27, there is only one point C' in the order $\{A'B'C'\}$ such that BC is congruent to $B'C'$.

XIII. *AB is congruent to BA.*

Proof. AB is transformed into BA by the point reflection whose center is the mid-point of AB.

XIV. *If A, B, C are three noncollinear points and D is a point in the order $\{BCD\}$, and if $A'B'C'$ are three noncollinear points and D' is a point in the order $\{B'C'D'\}$ such that the point pairs AB, BC, CA, BD are respectively congruent to $A'B'$, $B'C'$, $C'A'$, $B'D'$, then AD is congruent to $A'D'$.*

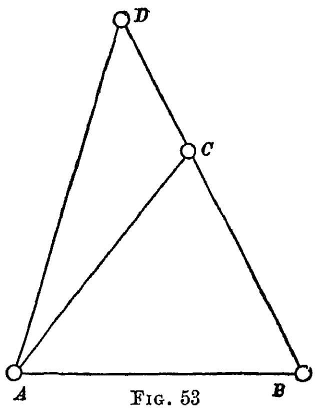


FIG. 53

Proof. Since AB is congruent to $A'B'$, there exists a displacement Δ which carries AB to $A'B'$. Let $\Delta(C)=C_1$, $\Delta(D)=D_1$. Also let C_2 and D_2 be the points to which C_1 and D_1 are transformed by the orthogonal line reflection having $A'B'$ as axis.

According to § 57, the pair BC is congruent to $B'C_1$ and to $B'C_2$; CA to C_1A' and C_2A'; BD to $B'D_1$ and $B'D_2$; and AD to $A'D_1$ and $A'D_2$. It follows that C' must coincide with C_1 or C_2, for otherwise there would be two circles, one with A' as center and the other with B' as center, containing the three points C_1, C_2, C'.

If $C'=C_1$, it follows, by Theorem 23, that $D'=D_1$, and hence that AD is congruent to $A'D'$. If $C'=C_2$, it follows, similarly, that $D'=D_2$, and hence that AD is congruent to $A'D'$.

DEFINITION. If O and X_0 are two points of a plane α, then the set of points $[X]$ of α such that OX is congruent to OX_0 is called a *circle.*

XV. *If the line joining the centers of two coplanar circles meets them in pairs of points, P_1Q_1 and P_2Q_2 respectively, such that $\{P_1P_2Q_1\}$ and $\{P_1Q_1Q_2\}$, the circles have two points in common, one on each side of the line joining the centers.*

Proof. This is the same as Theorem 29.

XVI. *If A, B, C are three points in the order $\{ABC\}$ and B_1, B_2, B_3, $\cdots$ are points in the order $\{ABB_1\}$, $\{AB_1B_2\}$, $\cdots$ such that AB is congruent to each of the point pairs BB_1, B_1B_2, $\cdots$, then there are not more than a finite number of the points B_1, B_2, $\cdots$ between A and C.*

Proof. Let B_∞ be the point at infinity of the line AB. Then B_1 is the harmonic conjugate of A with respect to B and B_∞, B_2 is the harmonic conjugate of B with respect to B_1 and B_∞; and so on. Thus $A, B, B_1, B_2, \cdots$ form a harmonic sequence of which B_∞ is the limit-point. Since C has a finite coördinate, the result follows from § 8, Chap. I.

The set of assumptions I–XVI is not categorical. It provides merely for the existence of such irrational points as are needed in constructions involving circles and lines (see § 77, below). It can be made categorical by adding Assumption XVII, § 29. It must be noted, however, that when XVII is added, X–XVI become redundant in the sense that it is possible to introduce ideal elements and then bring in the congruence relations by means of the definitions in this and the preceding chapters.

In order to convince himself that the assumptions given above are a sufficient basis for the theorems of Euclid, the reader should carry out the deduction from these assumptions of some of the fundamental theorems in Euclid's Elements. An outline of this process will be found in the monograph on the subject from which the assumptions have been quoted.*

In making a rigorous deduction of the theorems of elementary geometry, either from the assumptions above or from the general projective basis, it is necessary to derive a number of theorems which are not mentioned in Euclid or in most elementary texts. These are mainly theorems on order and continuity. They involve such matters as the subdivision of the plane into regions by means of curves, the areas of curvilinear figures, etc., all of which are fundamental in the applications of geometry to analysis, and vice versa. In so far as these theorems relate to circles, they have been partially treated in §§ 64–65 and will be further discussed in the next chapter. The methods used for the more general theorems on order and continuity, however, are less closely related to the elementary part of projective geometry and will therefore be postponed to a later chapter.

* Foundations of Geometry, by Oswald Veblen, in Monographs on Modern

67. Distance. In § 43 we have defined the magnitude of a vector OB as its ratio to a unit vector OA collinear with it; but in the affine geometry the magnitudes of noncollinear vectors are absolutely unrelated. In the parabolic metric geometry we introduce the additional requirement that any two unit vectors OA and $O'A'$ shall be such that the point pair OA is congruent to the point pair $O'A'$.

Thus, if a given unit vector OA is fixed and C^2 is the circle through A with O as center, any other unit vector must be expressible in the form Vect (OP), where P is a point of the circle. This gives two choices for the unit vector of any system of collinear vectors, and each of the two possible unit vectors is the negative of the other. Therefore, while it is possible under our convention to compare the absolute values of the magnitudes of noncollinear vectors, there is no relation at all between their algebraic signs. This corresponds to the fact that there is no unique relation between particular sense classes on two nonparallel lines.

Formulas in which the magnitudes of noncollinear vectors appear must, if they state theorems of the Euclidean geometry, be such that their meaning is unchanged when the unit vector on any line is replaced by its negative. This condition is satisfied, for example, in Exs. 2 and 4, § 71.

The ratio of two collinear vectors is invariant under the affine group; the magnitude of a vector is invariant under the group of translations; but the absolute value of the magnitude of a vector, according to our last convention, is invariant under the group of displacements. The last invariant may be defined directly in terms of point pairs as follows:

Definition. Let AB be an arbitrary pair of distinct points which shall be referred to as the *unit of distance.* If P and Q are any two points, let C be a point of the ray AB such that the pair AC is congruent to the pair PQ. The ratio

$$\frac{AC}{AB}$$

is called the *distance from P to Q,* and denoted by Dist (PQ). If L is any point and l any line, the distance from L to the foot of the

uniquely defined and positive whenever $P \neq Q$, and zero whenever $P = Q$. From the corresponding theorems on the magnitudes of vectors there follows the theorem that if $\{ABC\}$, then

$$\text{Dist}\,(AB) + \text{Dist}\,(BC) = \text{Dist}\,(AC).$$

Other properties of the distance-function are stated in the exercises.

The notion of the length (or circumference) of a circle may be defined as follows: Let $P_1, P_2, \cdots, P_n$ be n points in the order $\{P_1P_2 \cdots P_n\}$ on a circle, and let

$$p = \text{Dist}\,(P_1P_2) + \text{Dist}\,(P_2P_3) + \cdots + \text{Dist}\,(P_nP_1).$$

It can easily be proved that for a given circle C^2, the numbers p obtained from all possible ordered sets of points $P_1, P_2, \cdots, P_n$, for all values of n, do not exceed a certain number.

Definition. The number c, which is the smallest number larger than all values of p, is called the *length* or *circumference* of the circle C^2.

The proof of the existence of the number c will be omitted for the reasons explained below. The existence of c having been established, it follows without difficulty that if c and c' are the lengths of two circles with centers O and O', respectively, and passing through points P and P', respectively,

$$\frac{c}{c'} = \frac{\text{Dist}\,(OP)}{\text{Dist}\,(O'P')}.$$

Choosing the point pair $O'P'$ as the unit of distance and denoting the constant c' by 2π, this gives the formula

$$c = 2\pi \cdot \text{Dist}\,(OP). \tag{11}$$

The theory of the lengths of curves in general could be developed at the present stage without any essential difficulty. This subject, however, is very different (in respect to method, at least) from the other matters which we are considering, and therefore will be passed over with the remark that, starting with the theory of distance here developed, all the results of this branch of geometry may be obtained as applications of the integral calculus. Even the theory of the length of circles which we have summarized in the paragraphs above involves the ideas, if not the methods, of the calculus.

EXERCISES

1. Two point pairs AB and CD are congruent if and only if $\text{Dist}\,(AB) = \text{Dist}\,(CD)$.

2. If A, B, C are noncollinear points, $\text{Dist}\,(AB) + \text{Dist}\,(BC) > \text{Dist}\,(AC)$.

3. Two triangles ABC and $A'B'C'$ are similar in such a way that A corresponds to A', B to B', and C to C' if and only if

$$\frac{\text{Dist}\,(AB)}{\text{Dist}\,(A'B')} = \frac{\text{Dist}\,(AC)}{\text{Dist}\,(A'C')} = \frac{\text{Dist}\,(BC)}{\text{Dist}\,(B'C')}.$$

4. Relative to a coördinate system in which the axes are at right angles, the distance between two points (x_1, y_1), (x_2, y_2) is

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2},$$

the positive determination of the radical being taken. The distance from a point $(x_1 y_1)$ to a line $ax + by + c = 0$ is the numerical value of

$$\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}.$$

68. Area. The area of a triangle, as distinguished from the measure of an ordered point triad, may be defined as follows:

Definition. Relative to a unit triad OPQ (§ 49) such that the lines OP and OQ are orthogonal and the point pairs OP and OQ are congruent to the unit of distance, the positive number

$$\tfrac{1}{2}\,|\,m\,(ABC)\,|$$

is called the *area of the triangle ABC*, and denoted by a (

As was brought out in Chap. III, the theory of r
belongs properly to the affine geometry. Bu
for the area of a triangle in terms of base a
involves the ideas of distance and perpendicularity and hence belongs to the parabolic metric geometry. It should be noticed that this formula assumes that the side of the triangle which is regarded as the base does not pass through a double point of the absolute involution. This condition is satisfied under the hypotheses of §§ 63, 64, but is not always satisfied in a complex plane; whereas the definitions of equivalence and measure as given in Chap. III are entirely free of such restrictions.

The theory of areas in general depends on considerations of order and continuity which we have not yet developed, and which, like the theory of lengths of curves, belongs essentially to another branch of geometry than that with which we are concerned in this chapter. We shall, however, outline the definition of the area of an ellipse from the point of view of elementary geometry, because the derivation of the area of an ellipse from that of the circle affords rather an interesting application of one of the theorems about the affine group.

Let $P_1, P_2, \cdots, P_n$ be any finite number of points in the order $\{P_1 P_2 \cdots P_n\}$ on an ellipse E^2 with a point O as center, and let

$$A = a(OP_1P_2) + a(OP_2P_3) + \cdots + a(OP_nP_1).$$

It can easily be proved that there exists a finite number, $a(E^2)$, which is the smallest number which is greater than all values of A formed according to

DEFINITION. The number $a(E^2)$ is called the *area* of the ellipse.

In case E^2 is a circle, C^2, it is easy to prove that

$$a(C^2) = \pi r^2,$$

where π is the constant defined above and $r = \text{Dist}(OP_1)$.

Now suppose E^2 is an ellipse with two perpendicular conjugate diameters OA and OB which meet E^2 in A and B respectively, and let C^2 be the circle through A with O as center, and let C be the point in which the ray OB meets C^2. The homology Γ with OA as axis and the point at infinity of OB as center, which transforms B to C, is an affine transformation carrying the ellipse E^2 to the circle C^2. This homology transforms the triangle OAB to the triangle OAC; and the areas of these triangles satisfy the relation

$$\frac{a(OAC)}{a(OAB)} = \frac{\text{Dist}(OC)}{\text{Dist}(OB)} = k.$$

It follows, by § 50, that the homology transforms any triangle into one whose area is k times as large. By the definition of the area of an ellipse, therefore,

$$\frac{a(C^2)}{a(E^2)} = \frac{\text{Dist}(OC)}{\text{Dist}(OB)}.$$

Denoting Dist (OA) by a and Dist (OB) by b, this gives

$$a(E^2) = \frac{\pi a^2 b}{a} = \pi ab.$$

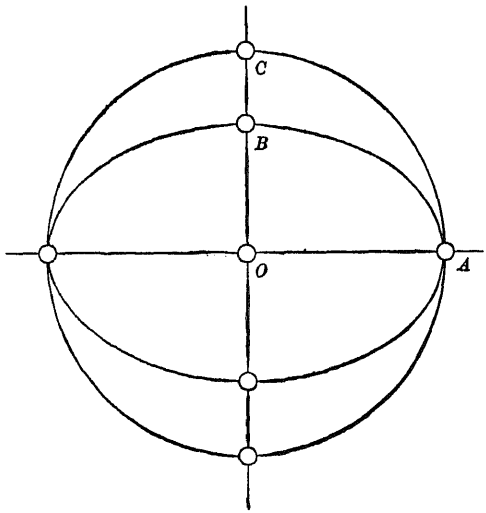


FIG. 54

EXERCISES

1. The numerical value of the measure of a point triad ABC is equal to $\text{Dist}(AB) \cdot \text{Dist}(CC')$, where C' is the foot of the perpendicular from C to the line AB.

2. If $abcd$ is a simple quadrilateral whose vertices are on a conic and P is a variable point of the conic,

$$\frac{\text{Dist}(Pa) \cdot \text{Dist}(Pc)}{\text{Dist}(Pb) \cdot \text{Dist}(Pd)}$$

is a constant (cf. Ex. 2, § 51).

3. If a projective collineation carries a variable point M and two fixed lines a, b to M', a', b' respectively, the number

$$\frac{\text{Dist}(Ma)}{\text{Dist}(Mb)} \div \frac{\text{Dist}(M'a')}{\text{Dist}(M'b')}$$

is a constant.

4. Let F be the center of a homology Γ and l the vanishing line, $\Gamma^{-1}(l_\infty)$. If P is a variable point and $Q = \Gamma(P)$,

$$\frac{\text{Dist}(FP)}{\text{Dist}(Pl)} = k \cdot \text{Dist}(FQ),$$

where k is a constant.

5. The area of an ellipse is $\pi a/2$, where a is the area of any inscribed parallelogram whose diagonals are conjugate diameters.

6. Among all simple quadrilaterals circumscribed to an ellipse, the ones whose sides are tangent at the ends* of conjugate diameters have the least area.

7. Among all simple quadrilaterals inscribed in an ellipse, the ones whose vertices are the ends of conjugate diameters have the greatest area.

8. Of all ellipses inscribed in a parallelogram, the one which has the lines joining the mid-points of opposite sides as a pair of conjugate diameters has the greatest area.

9. Of all ellipses circumscribed to a parallelogram, the smallest is the one having the diagonals as conjugate diameters.

69. The measure of angles. The unit of distance may be chosen arbitrarily, because any point pair can be transformed under the parabolic metric group into any other point pair. It is otherwise with angles or line pairs, because, for example, an orthogonal line pair cannot be transformed into a nonorthogonal pair. Therefore the systems of measurement for angles obtained by choosing different units are, in general, essentially different. We shall give an outline of the generally adopted system of measurement, basing it upon properties of the group of rotations leaving a point O invariant.

Let P_0 be an arbitrary point different from O, and C^2 the circle through P_0 with O as center. Let P_1 (fig. 55) be the point different from P_0 in which the line P_0O meets C^2, and let $P_{\frac{1}{2}}$ and $P_{\frac{3}{2}}$ be the points in which the perpendicular to P_0O at O meets C^2. By Cor. 1, Theorem 25, these points are in the order $\{P_0P_{\frac{1}{2}}P_1P_{\frac{3}{2}}\}$ on the circle. Let σ denote the segment $\overline{P_0P_{\frac{1}{2}}P_1}$. Any line through O meets C^2 in two points which are separated by P_0 and P_1, and hence meets σ in a unique point. Let $P_{\frac{1}{4}}$ be the point in which the line through O perpendicular to $P_0P_{\frac{1}{2}}$ meets σ. And, in general, let $[P_{\frac{1}{2^n}}]$, $n = 1, 2, \cdots$ be the set such that P is the point in which the line through O perpendicular to P_0P

The line $OP_{\frac{1}{4}}$ obviously meets the line $P_0P_{\frac{1}{2}}$ in the mid-point of the pair $P_0P_{\frac{1}{2}}$, and the mid-point is between P_0 and $P_{\frac{1}{2}}$. Hence, by Cor. 1, Theorem 25, we have the order relation $\{P'P_0P_{\frac{1}{4}}P_{\frac{1}{2}}\}$, where P' denotes, for the moment, the point not on σ in which the line $OP_{\frac{1}{4}}$ meets the circle. Since O is between $P_{\frac{1}{4}}$ and P', the same corollary gives $\{P_0P_{\frac{1}{4}}P_1P'\}$.

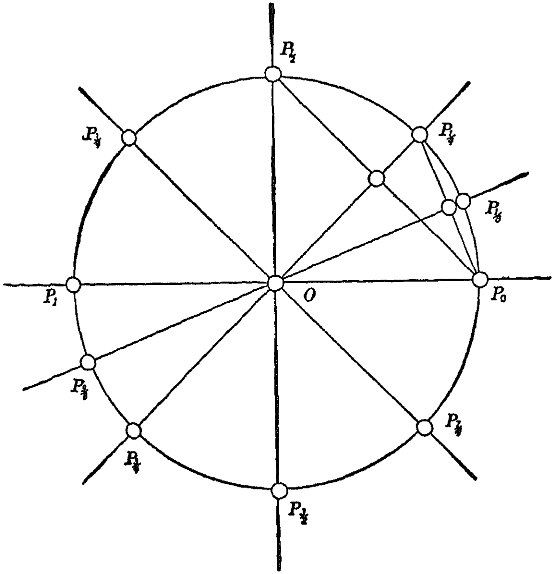


FIG. 55

Since $P_{\frac{1}{4}}$ is on the segment σ, we have either $\{P_0P_{\frac{1}{4}}P_{\frac{1}{2}}P_1\}$ or $\{P_0P_{\frac{1}{2}}P_{\frac{1}{4}}P_1\}$. The second of these alternatives, however, when combined with $\{P'P_0P_{\frac{1}{4}}P_{\frac{1}{2}}\}$, would imply $\{P'P_0P_1P_{\frac{1}{4}}P_{\frac{1}{2}}\}$, contrary to $\{P_0P_{\frac{1}{4}}P_1P'\}$. Hence $\{P_0P_{\frac{1}{2}}P_{\frac{1}{4}}P_1\}$ is impossible, and we must have $\{P_0P_{\frac{1}{4}}P_{\frac{1}{2}}P_1\}$. In like manner it is proved that $\{P_0P_{\frac{1}{8}}P_{\frac{1}{4}}P_1\}$ and, in general, that

$$\{P_0 \cdots P_{\frac{1}{2^n}}P_{\frac{1}{2^{n-1}}} \cdots P_{\frac{1}{2}}P_1\}.$$

Let Π denote the rotation (a point reflection in this case) which leaves O fixed and transforms P_0 to P_1, and let $\Pi^{\frac{1}{2^n}}$ denote the rotation transforming P_0 to $P_{\frac{1}{2^n}}$. The rotation $\Pi^{\frac{1}{2}}$, being the product of the orthogonal line reflection with $OP_{\frac{1}{4}}$ as axis followed by that with $OP_{\frac{1}{2}}$ as axis, carries the point pair $OP_{\frac{1}{2}}$ to the point pair OP_1. Hence*

$$(\Pi^{\frac{1}{2}})^2 = \Pi.$$

In like manner it follows that

$$(\Pi^{\frac{1}{2^n}})^2 = \Pi^{\frac{1}{2^{n-1}}}.$$

Now all rotations are direct (Theorem 26). Hence $S(P_0P_{\frac{1}{4}}P_{\frac{1}{2}}) = S(P_{\frac{1}{4}}P_{\frac{1}{2}}P_{\frac{3}{4}}) = S(P_{\frac{1}{2}}P_{\frac{3}{4}}P_1)$. Combining these relations with $\{P_0P_{\frac{1}{4}}P_{\frac{1}{2}}P_1\}$, we have the order relation $\{P_0P_{\frac{1}{4}}P_{\frac{1}{2}}P_{\frac{3}{4}}P_1\}$, and in general, by a like argument,

$$\{P_0P_{\frac{1}{2^n}}P_{\frac{2}{2^n}}P_{\frac{3}{2^n}}\cdots P_1\}.$$

Hence we have $\{P_0P_{\frac{m}{2^n}}P_{\frac{m'}{2^{n'}}}P_1\}$, whenever $0 < \frac{m}{2^n} < \frac{m'}{2^{n'}} < 1$, as can easily be seen on reducing the two fractions to a common denominator.

Since $\Pi^2 = 1$, it follows that whenever $m/2^n$ is expressible in the form $2k + \alpha$, k being an integer,

$$\Pi^{2k+\alpha} = \Pi^{\alpha} \quad \text{and} \quad P_{2k+\alpha} = P_{\alpha}. \tag{12}$$

Definition. Let π be the constant defined in § 67, (11). The number $\alpha \cdot \pi$, where $\alpha = m/2^n$, is called the *measure* of any angle congruent to $\measuredangle P_0OP_\alpha$. An angle whose measure is $\alpha\pi$ is also said *to be equal to* 2α *right angles.*

The measure of an angle is indeterminate according to this definition. In fact, according to (12), whenever the measure of an angle is β, it is also $2k\pi + \beta$, where k is any positive or negative integer. This indetermination can be removed by requiring that the measure β chosen for any angle shall always satisfy a condition of the form $0 \leqq \beta < 2\pi$, or $-\pi < \beta \leqq \pi$.

Since the rays $OP_{\frac{m}{2^n}}$ do not include all rays with O as center, the definition just given does not determine the measures of all angles. The required extension may be made by means of elementary continuity considerations, the details of which we shall omit. The essential steps required are: (1) to prove that if $\overline{P}$ be any point in the order $\{P_0\overline{P}P_{\frac{1}{2}}P_1\}$, there exists a positive integral value of n such that $\{P_0P_{\frac{1}{2^n}}\overline{P}P_{\frac{1}{2}}P_1\}$; (2) hence to prove that if P be any point on the circle not of the form $P_{\frac{m}{2^n}}$, the points of the form $P_{\frac{m}{2^n}}$ fall into two classes, $[P_\alpha]$ and $[P_\beta]$, such that $\{P_0P_\alpha PP_\beta\}$, and there is no point, except P, on every segment $\overline{P_\alpha PP_\beta}$ of the circle; (3) having required that $0 < \alpha < \beta < 2$, to define Π^{2k+x} (where k is an integer, positive, negative, or zero, and x is the number

such that $\alpha < x < \beta$ for all α's and β's) as the rotation about O carrying P_0 to P; (4) to show that if x is a rational number m/n, $(\Pi^x)^n = \Pi^m$; (5) to define measure of angle as above, but with the restriction that $\alpha = m/2^n$ removed; (6) to prove that the measure of the sum of two angles differs from the sum of the measures by $2k\pi$, the sum being defined as below.

DEFINITION. If a, b, c are any three rays having a common origin, but not necessarily distinct, any angle $\measuredangle a_1c_1$ congruent to $\measuredangle ac$ is said to be the *sum* of any two angles $\measuredangle a_2b_2$ and $\measuredangle b_3c_3$ such that $\measuredangle a_2b_2$ is congruent to $\measuredangle ab$ and $\measuredangle b_3c_3$ is congruent to $\measuredangle bc$. The sum $\measuredangle a_1c_1$ is denoted by $\measuredangle a_2b_2 + \measuredangle b_2c_2$.

For some purposes it is desirable to have a conception of angle according to which any two numbers are the measures of distinct angles. This may be obtained as follows:

DEFINITION. A ray associated with an integer, positive, negative, or zero, is called a *numbered ray*. An ordered pair of numbered rays having the same origin is called a *numbered angle*. If the measure of an angle $\measuredangle hk$ in the earlier sense is a, where $0 \leqq a < 2\pi$, the measure of a numbered angle in which h is associated with m, and k with n, is

$$2(n-m)\pi + a.$$

Defining the sum of two numbered angles in an obvious way, it is clear that the sum of two numbered angles has a measure which is the sum of their measures.

The trigonometric functions can now be defined, following the elementary textbooks, as the ratios of certain distances multiplied by ± 1 according to appropriate conventions. This we shall take for granted in the future as having been carried out.

70. The complex plane. Instead of the assumption in § 64, we could assume that the Euclidean plane is obtained by leaving out one line from the complex projective plane (A, E, J, or A, E, H, C, $\overline{\text{R}}$, I). All the results of Chap. III and of the present chapter up to § 63 are applicable to this case. The rest of the theory, however, is essentially different from that of the real plane, because the absolute involution necessarily has two double points and because a line does not satisfy the one-dimensional order relations. Thus the minimal lines play a principal rôle and must be regarded as exceptional in the statement of a large class of theorems; and another large class of theorems of elementary geometry (those involving order relations) disappears entirely.

For the present, therefore, we shall confine attention to the geometry of reals, but shall make use, whenever we find it convenient to do so, of the fact (§ 6) that a real space S may be regarded as immersed in a complex space, S′, in such a way that every line l of S is contained in a unique line l' of S′. As a direct consequence it follows that any conic C^2 of S is a subset of the points of a unique conic of S′. For any five points of C^2, regarded as points of S′, determine a unique conic of S′ which, by construction (§ 41, Vol. I), contains all points of C^2 and is uniquely determined by any five of its points. Similar reasoning will show that any plane π of S is contained in a unique plane π' of S′; and like remarks may be made with regard to any one-, two-, or three-dimensional form.

A like situation arises with respect to transformations. A projective transformation Π of a form in S is fully determined, according to the fundamental theorem of projective geometry, by its effect on a finite set* of elements of S. Since the fundamental theorem is also valid in S′, there is a unique projective transformation Π' which has the same effect on this set of elements as Π.

Specializing these remarks somewhat we have: A Euclidean plane π of S is a subset of the points of a certain Euclidean plane π' of S′. The line at infinity l_∞ associated with π is a subset of the line at infinity l'_∞ associated with π'. The absolute involution I on l_∞ determines an involution I′ on l'_∞ in which all the pairs of I are paired. The involution I′ has two imaginary double points, the circular points (§ 56), which shall be denoted by I_1 and I_2. Since a circle in π is a conic having I as an involution of conjugate points, every circle in π is a subset of the points on a conic in π' which passes through I_1 and I_2.

The problem of the intersection of a line and a circle, or indeed of a line and any ellipse, can now be discussed completely. In the proof of Theorem 25 the intersection of a line l and an ellipse E^2 was seen to depend on finding the double points of a certain projectivity $[L_1] \overline{\wedge} [L_2]$ on l. Any three points L_1', L_1'', L_1''', and their correspondents L_2', L_2'', L_2''', determine a projectivity on the complex line l' containing l, and, by the fundamental theorem of projective geometry, this projectivity is identical with $[L_1] \overline{\wedge} [L_2]$ so far as real points are concerned. The double points of this projectivity are common to the complex

line containing l and the complex conic containing E^2. These points are real if the hypothesis of Theorem 25 is satisfied; they are real and coincident if l is tangent to E^2; otherwise they are imaginary.

A similar discussion will be made in the next section of the problem of the intersection of two circles, but first let us make certain definitions and conventions which will simplify our terminology.

According to the definitions in § 6, any point of S' is said to be *complex*, and a complex point is *real* or *imaginary* according as it is contained in S or not. In the case of lines, however, we have three things to distinguish: a line of the space S, a line of S' which contains a line of S as a subset, and a line of S' which contains no such subset. In current usage a line of the last sort is called *imaginary*, a line of either of the first two sorts is called *real*, and a line of either of the last two sorts is called *complex*. The current terminology therefore permits a confusion between a real line as a locus in S and a real line as a particular kind of a complex line.

In most cases, however, no misunderstanding need be caused by this ambiguity of language, and we shall in future usually employ the same notation for the real line l of S and the line l' of S' which contains l. The same remarks apply to conic sections and, indeed, to all one-dimensional forms.

DEFINITION. Any element (point, line, or plane) or set of elements of S' is said to be *complex*. Any element or set of elements of S is said to be *real*. A line or plane of S' which contains a line or plane, respectively, of S is said to be a *real line* or *real plane of* S'. A one-dimensional form of S', a subset of whose elements are real elements of S' and contain all the elements of a one-dimensional form of S, is called a *real one-dimensional form of* S'. An element or one-dimensional form of S' which is not a real element or real one-dimensional form of S' is said to be *imaginary*.

DEFINITION. A projective transformation of a real form of S' is said to be *real* if it transforms each real element of S' into a real element of S'.

Strictly speaking, these definitions distinguish between the two senses of the word "real" by phrases such as "real line of S'." But in practice we shall drop the "of S'." The one-dimensional forms as thus far defined are all of the first or second degrees, but the defini-
tion can be extended without essential modification to forms of higher

are really the double points of I′ will be referred to in future as the double points of the absolute involution I. In like manner, any line l and circle C^2 which have no real points in common will be said to have in common the two points common to the complex line and the complex conic which contain l and C^2 respectively.

The utility of these conventions will be understood by the reader if he will write out in full the discussion of pencils of circles in the following section, putting in explicitly, in notation and language, the distinction between elements of S and S′.

It is also convenient in many cases to extend the formulas for distance, area, etc. given in §§ 67–69 to imaginary elements. Thus, for example, in case (x_1, y_1) and (x_2, y_2) are imaginary points such that $(x_1 - x_2)^2 + (y_1 - y_2)^2$ is a positive real number, $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ will be referred to as the distance from (x_1, y_1) to (x_2, y_2). Extensions of terminology of this self-evident sort will be made when needed, without further explanation.

71. Pencils of circles. Consider two circles C_1^2 and C_2^2 in a real Euclidean plane. Let their centers be denoted by C_1 and C_2, and in case $C_1 \neq C_2$, let b denote the line C_1C_2. By Theorem 25, b meets each circle in a pair of real points which we shall denote by P_1Q_1 and P_2Q_2 respectively. The two pairs may be entirely distinct, in which case let Γ denote the involution on b transforming each pair into itself; or they may have one point in common, in which case the line through this point perpendicular to b is a common tangent of the two circles. The two pairs cannot coincide, because the circles would then coincide. Thus four cases may be distinguished:

(1) The circles have the same center.
(2) The circles have a common tangent and point of contact.
(3) The involution Γ is direct.
(4) The involution Γ is opposite.

A circle is, by § 60, a real conic which, according to the terminology of the last section, contains the double points of the absolute involution. Let us denote these points (the circular points) by I_1 and I_2 and apply the results of § 47, Vol. I, on pencils of conics.

The lines OI_1 and OI_2 are then tangent to both circles at I_1 and I_2 respectively. Hence, by reference to § 47, Vol. I, it is evident that the two circles belong to a pencil of circles of Type IV.

In the second case C_1^2 and C_2^2 have in common the points I_1 and I_2 as well as a common tangent and point of contact. Hence they belong to a pencil of Type II which contains all circles touching* the given line at the given point.

In the third case, since the involution Γ is direct, the pairs P_1Q_1 and P_2Q_2 separate each other. Hence, by Theorem 29, the circles have two real points, A_1 and A_2, in common. Hence they belong to a pencil of Type I consisting of all conics through A_1, A_2, I_1, and I_2. This may also be seen as follows:

Since the involution Γ has no double points (§ 21), it has a center (§ 43) which we shall call O. Let a be the line perpendicular to b at O. Then by the argument used in the proof of Theorem 29, O is between P_1 and Q_1. Hence a meets C_1^2 in two real points A_1 and A_2 (fig. 52). The pencil of conics through A_1, A_2, I_1, I_2 meets b in the pairs of an involution among which are P_1Q_1 and O and the point at infinity of b. Hence C_2^2 is a conic of the pencil, and hence a meets C_2^2 in A_1 and A_2. In this case, therefore, the two circles belong to a pencil of Type I.

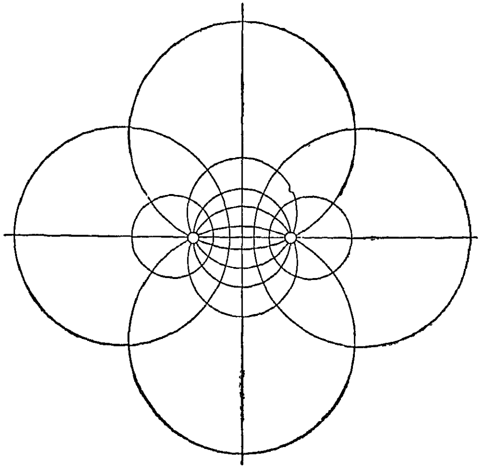


FIG. 56

In the fourth case the involution Γ cannot have a double point at infinity, because then the other double point would have to be the mid-point of P_1Q_1 and also of P_2Q_2, and thus C_1^2 and C_2^2 would have a common center. Hence in this case also the center O of the involution Γ is an ordinary point. Let a denote

*A conic and one of its tangent lines are said to *touch* each other at the point of contact. Two conics touching a line at the same point are said to *touch* each other.

the perpendicular to b at O, and let A_1 and A_2 be the points in which a meets C_1^2. These points are imaginary; for otherwise, since they are interchanged by the orthogonal line reflection with b as axis, O would be between them, and hence, by Cor. 1, Theorem 25, O would be between P_1 and Q_1, contrary to the hypothesis that Γ is opposite. Precisely as in the third case it follows that A_1 and A_2 are also on C_2^2. Hence in this case also C_1^2 and C_2^2 belong to a pencil of Type I.

In each case the facts established make it clear that the two circles could not both be members of more than one pencil of conics. Since any two circles fall under one of the four cases, we have

THEOREM 30. DEFINITION. *Any circle contains the real points of a certain conic in the complex plane. Two conics determined by circles are contained in a unique pencil of conics, which is of Type I, II, or IV. The set of circles which the conics of such a pencil have in common with the real plane is called a* pencil of circles. *If the pencil of conics is of Type IV, the pencil of circles is the set of all circles having a fixed point as center; if the pencil of conics is of Type II, the pencil of circles is the set of all circles tangent to a given line at a given point; if the pencil of conics is of Type I, the pencil of circles is the set of all circles having a given pair of distinct real points in common, or else the set of all circles with centers on a given line and meeting this line in the pairs of an involution with two ordinary double points.*

DEFINITION. The line a joining the centers of two nonconcentric circles is called *the line of centers* of the two circles or of the pencil of circles which contains them. If the circles have a common tangent and point of contact, this tangent is called the *radical axis* of the two circles or of the pencil of circles; if not, the line perpendicular to a at the center of the involution in which the circles of the pencil meet a is called the *radical axis.* The double points of this involution are called the *limiting points* of the pencil of circles. Any circle of the pencil is said to be *about* either one, or both, of the limiting points.

The discussion above has established

THEOREM 31. *The radical axis of two circles passes through all points common to them which are not on the line at infinity. The limiting points of the pencil which they determine are real if the circles meet only in imaginary points and imaginary if they meet in two real points.*

THEOREM 32. *The circular points, the limiting points of a pencil of circles of Type I, and the two points not at infinity in which the circles of the pencil intersect are the pairs of opposite vertices of a complete quadrilateral. The sides of the diagonal triangle of this quadrilateral are l_∞, the radical axis, and the line of centers of the pencil.*

Proof. Let A_1 and A_2 (fig. 57*) be the points other than I_1 and I_2 common to the circles of the pencil, and let B_1 and B_2 be the points of intersection of the pairs of lines I_1A_1, I_2A_2 and I_1A_2, I_2A_1 respectively. Whether A_1 and A_2 are real or imaginary, the line $A_1A_2 = a$, which is the radical axis, is real. Hence its point at infinity A_∞ is real; and hence the line B_1B_2, the polar of A_∞ with regard to any circle of the pencil, is real.

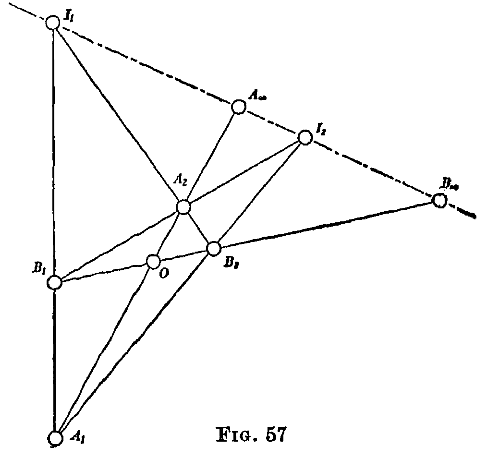


FIG. 57

Since the line $b = B_1B_2$ is the polar of A_∞, it contains the centers of all conics through A_1, A_2, I_1, I_2. Hence b is the line of centers of the pencil of circles through A_1 and A_2. The points B_1 and B_2 being diagonal points of the complete quadrangle $A_1A_2I_1I_2$ are evidently the double points of the involution in which the pencil of circles meets b, and hence are the limiting points of the pencil.

Taking Theorems 31 and 32 together, we see that any pair of real points A_1, A_2 determines a pair of imaginary points B_1, B_2 such that either pair is the pair of limiting points of the pencil of circles through the other pair; that, conversely, any pair of imaginary points B_1, B_2, which are common to two circles, determines two real points A_1, A_2 which are in the above relation to B_1, B_2; and that the three pairs A_1A_2, B_1B_2, I_1I_2 are pairs of opposite vertices of a complete quadrilateral. The relation between the two pencils of circles, the one

symmetrical. It can be described in purely real terms by means of the following theorems and definition:

THEOREM 33. DEFINITION. *If two circles have a point in common such that the tangents to the two circles at this point are orthogonal, the two circles have another such point in common. Two circles so related are said to be* orthogonal *to each other.*

Proof. An orthogonal line reflection whose axis is the line of centers transforms each circle into itself and transforms the given point of intersection into another point of intersection. Since orthogonal lines are transformed to orthogonal lines, the tangents at the second point are also orthogonal.

THEOREM 34. *If a line through the center of a circle C^2 meets the circle in a pair of points P_1Q_1 and meets any orthogonal circle K^2 in a pair of points P_2Q_2, the pairs P_1Q_1 and P_2Q_2 separate each other harmonically. Conversely, if P_1Q_1 and P_2Q_2 separate each other harmonically, any circle through P_2 and Q_2 is orthogonal to C^2.*

Proof. Let T be one of the points common to the two circles, and let t be the tangent to the circle TP_2Q_2 at T. The pencil of circles tangent to t at T meets the line P_1P_2 in the pairs of an involution Γ, and hence the first statement of the theorem will follow if we can prove that P_1 and Q_1 are the double points of this involution. The line perpendicular to t at T and the line perpendicular to P_1P_2 at P_1 are tangents to the circle TP_1Q_1 at T and P_1 respectively, and hence (Ex. 4, § 60) meet in a point M such that the pairs MP_1 and MT are congruent. Hence the circle through T with M as center is tangent to t at T and to P_1P_2 at P_1. Hence P_1 is a double point of Γ. A similar argument shows that Q_1 is also a double point.

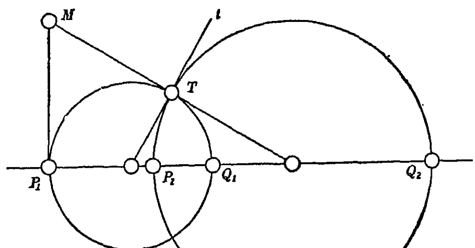


FIG. 58

To prove the converse proposition we observe that there is only one circle through P_2 and T and orthogonal to C^2. One such circle, by

the argument above, passes through the point Q_2, which is harmonically separated from P_2 by P_1 and Q_1. Hence the circle P_2Q_2T is orthogonal to C^2.

As a corollary we have

COROLLARY 1. *The set of all circles orthogonal to a pencil of Type I is the pencil of circles through the limiting points of the first pencil.*

Another form in which this result may be stated is the following:

COROLLARY 2. *Let C^2 be a circle, A_1 any point not its center, and A_2 the point on the line joining A_1 to the center of C^2 which is conjugate to A_1 with regard to the conic C^2. Then all circles through A_1 and orthogonal to C^2 meet in A_2.*

DEFINITION. Two points are said to be *inverse* with respect to a circle if and only if they are conjugate with regard to the circle and collinear with its center. The transformation by which every point corresponds to its inverse is called an *inversion* or a *transformation by reciprocal radii.*

Thus the center of the circle is inverse to every real point at infinity. We shall return to the study of inversions in a later chapter.

EXERCISES

1. In case the limiting points of a pencil of circles are real, the radical axis is their perpendicular bisector.

2. If O is any point of the plane of a circle, and a variable line through O meets the circle in two points X, Y, the product $OX \cdot OY$ is constant, and equal to $(OT)^2$ in case there is a line OT tangent to the circle at T. The product $OX \cdot OY$ is called the *power* of O with respect to the circle.

3. The power of any point of the radical axis of a pencil of circles with respect to all circles of the pencil is a constant, and this constant is the same for all points of the radical axis.

4. If O is the center of a circle, C any point of the circle, and A_1 and A_2 any two points inverse with respect to it,

$$OA_1 \cdot OA_2 = (OC)^2.$$

5. Through two points not inverse relative to a given circle, there is one and but one circle orthogonal to it.

6. By a *center of similitude* of two circles is meant the center of a dilation (§ 47) or translation which transforms one of the circles into the other. If the circles are concentric, they have one center of similitude; if they are not concentric, they have two. The centers of similitude harmonically separate the

circles is called the *interior*, and the other is called the *exterior*, center of similitude. The common tangents of two circles meet in the centers of similitude.

7. Three circles whose centers are not collinear determine by pairs six centers of similitude which are the vertices of a complete quadrilateral having the centers of the circles as vertices of its diagonal triangle. Generalize to the case of n circles.

8. If a circle K^2 meets two circles C_1^2 and C_2^2 in four points at which the pairs of tangents are congruent or symmetric, the four points are collinear by pairs with the centers of similitude of C_1^2 and C_2^2. Prove the converse proposition.

72. Measure of line pairs. The circular points I_1, I_2 figure in a very important formula for the measure of a pair of lines.* With the exception of these two points, and two lines i_1, i_2 which pass through them, all the points and lines to which we shall refer in this section are real.

The center and the point at infinity of the axis of an orthogonal line reflection are harmonically conjugate with regard to I_1 and I_2. Hence any orthogonal line reflection, regarded as a transformation of the complex space, interchanges I_1 and I_2, and any displacement leaves I_1 and I_2 separately invariant. Moreover, there exists a displacement transforming any (real) point of l_∞ to any other (real) point of l_∞. Hence a necessary and sufficient condition that a pair of points P, P' of l_∞ be transformable by a displacement to a pair Q, Q' of l_∞ is

$$\text{(13)} \qquad \text{R}(PP', I_1I_2) = \text{R}(QQ', I_1I_2).$$

Now any pair of lines meeting l_∞ in P and P' can be transformed by a translation into any other pair of lines meeting it in P and P', and any pair of lines meeting l_∞ in Q and Q' can be transformed by a translation into any other pair of lines meeting it in Q and Q'. Hence the necessary and sufficient condition that a pair of lines meeting l_∞ in P and P' be congruent to a pair of lines meeting it in Q and Q' is (13).

This suggests as a possible definition of the measure of a pair of nonparallel lines l_1, l_2,

$$\text{R}(l_1l_2, i_1i_2),$$

where i_1 and i_2 are the lines joining the point of intersection of l_1 and l_2 to I_1 and I_2 respectively. It would satisfy the requirement of

*This formula is due to A. Cayley. Cf. Encyclopädie der Math. Wiss. III *AB* 9,

being unaltered by displacements. In the case of measure of point pairs, however, we have

$$\mathrm{Dist}\,(AB) + \mathrm{Dist}\,(BC) = \mathrm{Dist}\,(AC)$$

whenever $\{ABC\}$, and this condition is not satisfied by the cross ratio given above. We have, in fact,

$$\mathrm{R}\,(l_1 l_2,\, i_1 i_2) \cdot \mathrm{R}\,(l_2 l_3,\, i_1 i_2) = \mathrm{R}\,(l_1 l_3,\, i_1 i_2) \tag{14}$$

whenever l_1, l_2, l_3 are concurrent. This is easily verified by substituting in the formula for cross ratio (§ 56, Vol. I).

From (14) it is obvious that if we define

$$m\,(l_1 l_2) = c \log \mathrm{R}\,(l_1 l_2,\, i_1 i_2), \tag{15}$$

the measure of line pairs will satisfy the condition

$$m\,(l_1 l_2) + m\,(l_2 l_3) = m\,(l_1 l_3)$$

whenever l_1, l_2, l_3 are concurrent. Since the logarithm is a multiple-valued function, we must specify which value is chosen; and we must also determine the constant c conveniently.

Making use of the same coördinate system as in § 62, any point on l_∞ may be denoted by $(0, \alpha, \beta)$. In case α/β is real, $(\alpha/\beta)^2 > 0$, and hence α and β may be multiplied by a factor of proportionality so that

$$\alpha^2 + \beta^2 = 1. \tag{16}$$

Throughout the rest of this section we shall suppose α and β subjected to this condition. This is equivalent to supposing that

$$\alpha = \cos\,(\theta + 2\,n\pi), \quad \beta = \sin\,(\theta + 2\,n\pi),$$

where $0 \leqq \theta \leqq 2\,\pi$, and n is an integer, positive, negative, or zero.

The double points of the absolute involution satisfy the condition (§ 62)

$$\alpha^2 + \beta^2 = 0,$$

and so may be written

$$I_1 = (0,\ 1,\ i) \quad \text{and} \quad I_2 = (0,\ 1,\ -i),$$

where $i = \sqrt{-1}$. Now if l_1 and l_2 meet l_∞ in $(0, \alpha_1, \beta_1)$ and $(0, \alpha_2, \beta_2)$ respectively, it follows that (§ 58, Vol. I)

$$\mathrm{R}\,(l_1 l_2,\, i_1 i_2) = \frac{\alpha_1 - i\beta_1}{\alpha_1 + i\beta_1} \div \frac{\alpha_2 - i\beta_2}{\alpha_2 + i\beta_2}$$

$$(\alpha_1\alpha_2 + \beta_1\beta_2) + i\,(\alpha_2\beta_1 - \alpha_1\beta_2)$$

The numbers $\alpha = \alpha_1\alpha_2 + \beta_1\beta_2$ and $\beta = \alpha_1\beta_2 - \alpha_2\beta_1$ satisfy the condition $\alpha^2 + \beta^2 = 1$. In fact, if $\alpha_1 = \cos\theta_1$ and $\alpha_2 = \cos\theta_2$, then $\alpha = \cos\theta$ and $\beta = \sin\theta$, where $\theta = \theta_1 - \theta_2 + 2n\pi$. Hence

$$\mathrm{R}(l_1l_2,\ i_1i_2) = \alpha^2 - \beta^2 + 2i\alpha\beta.$$

Here again, $\overline{\alpha} = \alpha^2 - \beta^2$ and $\overline{\beta} = 2\alpha\beta$ satisfy the condition

$$\overline{\alpha}^2 + \overline{\beta}^2 = 1.$$

In fact, $\overline{\alpha} = \cos 2\theta$. Thus

$$(17) \qquad \begin{aligned}\mathrm{R}(l_1l_2,\ i_1i_2) &= \overline{\alpha} + i\overline{\beta}\\ &= \cos 2\theta + i\sin 2\theta\\ &= e^{2i\theta}.\end{aligned}$$

Hence

$$(18) \qquad \log \mathrm{R}(l_1l_2,\ i_1i_2) = 2i\theta,$$

where 2θ is real and may be chosen so that $0 \leqq 2\theta < 2\pi$. Hence, choosing the constant c in (15) as $\frac{-i}{2}$, we have

$$(19) \qquad m(l_1l_2) = \frac{-i}{2}\log \mathrm{R}(l_1l_2,\ i_1i_2) = \theta,$$

where θ may be chosen so that $0 \leqq \theta < \pi$.

The formula (19) is interesting in connection with the theorem that the sum of the angles of a triangle is equal to two right angles. This proposition can easily be established without the consideration of imaginaries, on the basis of the definitions in the last section. From our present point of view, however, it appears as follows: Let the three sides of a triangle be a, b, c, and let them meet the line at infinity in A_∞, B_∞, C_∞ respectively. It is easily verifiable that

$$\mathrm{R}(A_\infty B_\infty,\ I_1I_2)\cdot\mathrm{R}(B_\infty C_\infty,\ I_1I_2)\cdot\mathrm{R}(C_\infty A_\infty,\ I_1I_2) = 1,$$

from which it follows by (19) that

$$m(ab) + m(bc) + m(ca) = \pi.$$

Here we have a theorem on the line pairs rather than on the angles of a triangle. Indeed, (19) is necessarily a formula for the measure of a pair of lines and not of an angle, because of the fact that two opposite rays determine the same point at infinity.

The number $m(ab)$ may also be defined as the smallest value between 0 and 2π, inclusive, of the measures of the four angles $\measuredangle a_1b_1$ which may be formed by a ray a_1 of a and a ray b_1 of b.

Following the common usage, we shall say that two pairs of lines which are congruent *make equal angles*, etc.

EXERCISES

1. If A and B are any two points, the locus of a point P such that the rays PA and PB make a constant angle is a circle.

2. If in two projective flat pencils three lines of one make equal angles with the corresponding three lines of the other, the angle between any two lines of the one is the same as the angle between the corresponding lines of the other.

3. If OA, OB, OC, OD are four lines of a flat pencil,

$$\text{R}(OA, OB;\ OC, OD) = \frac{\sin \measuredangle AOC}{\sin \measuredangle AOD} \div \frac{\sin \measuredangle BOC}{\sin \measuredangle BOD}.$$

In case the four lines form a harmonic set,

$$2 \cot \measuredangle AOB = \cot \measuredangle AOC + \cot \measuredangle AOD.$$

4. If A_1, A_2, A_3, A_4 are four points of a circle,

$$A_1A_3 \cdot A_2A_4 = A_1A_2 \cdot A_3A_4 + A_1A_4 \cdot A_2A_3,$$

where A_iA_j represents Dist (A_iA_j) or $-$ Dist (A_iA_j) according as $S(OA_iA_j) = S(OA_1A_2)$ or not, O being an arbitrary point of the circle and $S(OA_iA_j)$ being a sense-class on the circle.

5. If a, b, c are the sides of a triangle and a_1a_2, b_1b_2, c_1c_2 are pairs of lines through the vertices bc, ca, ab respectively, the six lines a_1, a_2, b_1, b_2, c_1, c_2 are tangents of a conic if and only if

$$\frac{\sin(a_1b)}{\sin(a_1c)} \cdot \frac{\sin(a_2b)}{\sin(a_2c)} \cdot \frac{\sin(b_1c)}{\sin(b_1a)} \cdot \frac{\sin(b_2c)}{\sin(b_2a)} \cdot \frac{\sin(c_1a)}{\sin(c_1b)} \cdot \frac{\sin(c_2a)}{\sin(c_2b)} = 1.$$

6. The points of a ray having (x, y) as origin may be represented in the form

$$(x + \lambda\alpha,\ y + \lambda\beta),$$

where α and β are fixed and $\lambda > 0$. There is a one-to-one reciprocal correspondence between the rays having (x, y) as origin and the ordered pairs of values of α and β which satisfy the condition

$$\alpha^2 + \beta^2 = 1.$$

When α and β satisfy this condition, the numerical value of λ is the distance between (x, y) and $(x + \lambda\alpha, y + \lambda\beta)$.

7. Two angles formed by the pairs of rays

$$\begin{matrix} (x_0 + \lambda\alpha,\ y_0 + \lambda\beta) \text{ and } (x_0 + \lambda\alpha',\ y_0 + \lambda\beta'), \\ (\overline{x}_0 + \lambda\overline{\alpha},\ y_0 + \lambda\overline{\beta}) \text{ and } (\overline{x}_0 + \lambda\overline{\alpha}',\ \overline{y_0} + \lambda\overline{\beta}') \end{matrix} \qquad \lambda > 0$$

respectively are congruent if and only if

$$\alpha\alpha' + \beta\beta' = \overline{\alpha}\overline{\alpha}' + \overline{\beta}\overline{\beta}'.$$

8. Relative to the homogeneous coördinates employed above, the formula for the distance between (x_0, x_1, x_2) and (y_0, y_1, y_2) may be written

$$\frac{\sqrt{(x_1y_0 - x_0y_1)^2 + (x_2y_0 - x_0y_2)^2}}{x_0y_0} = \frac{1}{x_0y_0} \begin{vmatrix} x_0 & x_1 & x_2 \\ y_0 & y_1 & y_2 \end{vmatrix}^{\frac{1}{2}} \cdot \begin{vmatrix} x_0 & x_1 & x_2 \\ y_0 & y_1 & y_2 \end{vmatrix}^{\frac{1}{2}} \cdot$$

73. Generalization by projection. The relation established in § 66 between Euclidean and projective geometry furnishes a source of new theorems in each. A theorem which has been proved for projective geometry can be specialized into a theorem of Euclidean geometry, or a theorem of Euclidean geometry may be generalized so as to furnish a theorem of projective geometry.

The two processes, of generalization and of specialization, may often be combined in a happy way with the principle of duality or with other general methods of projective geometry. Thus a theorem proved for Euclidean geometry can be generalized into a theorem of projective geometry and the dual of the general theorem specialized into a new theorem of Euclidean geometry. As an example, let us take the theorem of Euclid:

A. *The perpendiculars from the vertices of a triangle to the opposite sides meet in a point* (the orthocenter).

The sides of the triangle meet the line at infinity in three points, and the three perpendiculars are lines from the vertices to the conjugates of these three points in the absolute involution. The Euclidean theorem is therefore a special case of the following projective theorem:

B. *The lines joining the vertices of a triangle to the conjugates, with respect to an arbitrary elliptic involution on a line l, of the points in which the opposite sides meet l, are concurrent.*

This is a portion of Theorem 27, Chap. IV, Vol. I, the orthocenter and the three vertices of the triangle being the vertices of a complete quadrangle. But though the Euclidean theorem is a special case, yet the general theorem for elliptic involutions in real geometry may easily be proved by means of it. For, given any elliptic involution whatever and any triangle, the involution can be projected into the absolute involution and the given triangle will go into a triangle of the Euclid-

transformation which carried the involution into the absolute involution would carry the triangle into one whose sides are not all real.

Now consider the plane dual of the projective theorem, B.

B$'$. *The points of intersection of the sides of a triangle with the conjugates in an arbitrary involution at a point L, of the lines joining the vertices to L, are collinear.*

If the involution at L is taken as the orthogonal involution we have the Euclidean theorem:

A$'$. *The three sides of a triangle are met in three collinear points by the perpendiculars from a fixed point to the lines joining this point to the opposite vertices.*

The second of the two processes which we are here emphasizing, namely the discovery of Euclidean theorems by specializing projective ones, is brilliantly illustrated in many of the textbooks on projective geometry. We may mention the following:

L. Cremona, Elements of Projective Geometry, Oxford, 1894.
T. Reye, Geometrie der Lage, Leipzig, 1907–1910.
R. Sturm, Die Lehre von den Geometrischen Verwandtschaften, Leipzig, 1909.
R. Böger, Geometrie der Lage, Leipzig, 1900.
H. Grassman, Projective Geometrie der Ebene, Leipzig, 1909.
J. J. Milne, Cross-Ratio Geometry, Cambridge, 1911.
J. L. S. Hatton, Principles of Projective Geometry, Cambridge, 1913.

The reader will find material for the illustration of the second process, namely the discovery of projective theorems by generalizing metric ones, in Euclid's Elements, and even more in such books as the following:

J. Casey, A Sequel to the First Six Books of the Elements of Euclid, Dublin, 1888.
C. Taylor, Ancient and Modern Geometry of Conics, Cambridge, 1881.
J. W. Russell, Elementary Treatise on Pure Geometry, Oxford, 1905.

The class of theorems which are here in question will be dealt with to some extent in the following chapter, and the methods available will be extended in Chap. VI by the study of inversions. But on account of the magnitude of the subject many important theorems will be found relegated to the exercises and many others omitted entirely. In nearly every such case, however, a good treatment can be found in one or another of the books on projective geometry referred to above.

The current textbooks do not often classify theorems on the basis of the geometries to which they belong (§ 34) and the assumptions which are necessary for their proof (§ 17). Some progress has been made on such a classifi-

Another criticism on current books is that they employ imaginary points in a rather shy and awkward manner. This is doubtless due to the fact that, previous to a logical treatment of the subject based on definite assumptions, the geometry of reals was regarded as having, somehow, a higher degree of validity than the complex geometry. The reader will often find it easy to abbreviate the proofs of theorems in the literature by a free use of imaginary elements (cf. § 78).

EXERCISES

1. Generalize projectively the following theorems:

(*a*) The medians of a triangle meet in a point.

(*b*) The perpendiculars at the mid-points of the sides of a triangle meet in a point.

(*c*) The diagonals of a parallelogram bisect each other.

2. Let A_1, B_1, C_1 be the points in which the lines joining the vertices A, B, C, respectively, of a triangle to the orthocenter, O, meet the opposite sides. The circle through A_1, B_1 and C_1 contains the mid-points of the pairs AB, BC, CA and of the pairs OA, OB, OC. This circle is called the *nine point* or *Feuerbach circle* of the triangle. Cf. Ex. 7, § 41.

3. A hyperbola whose asymptotes are orthogonal is said to be *equilateral* or *rectangular*. Every hyperbola passing through four points of intersection of two equilateral hyperbolas is an equilateral hyperbola.

4. All equilateral hyperbolas circumscribed to a triangle pass through its orthocenter.

5. The centers of the equilateral hyperbolas circumscribed to a triangle lie on the nine-point circle.

CHAPTER V *

ORDINAL AND METRIC PROPERTIES OF CONICS

74. One-dimensional projectivities. The general discussion of one-dimensional projectivities in Chap. VIII, Vol. I, has a great many points of contact with the ordinal and metric theorems of the last three chapters. For example, a rotation leaving a point O invariant transforms into itself any circle C^2 with O as a center. The transformation effected on the circle by the rotation is a one-dimensional projectivity having the point O as center and the line at infinity as axis. The defining property of the axis of the projectivity in this case is that if a pair of points AB of the circle be rotated into a pair $A'B'$ (i.e. if $\measuredangle AOB$ be congruent to $\measuredangle A'OB'$), then the line AB' is parallel to the line $A'B$, which is a well-known Euclidean theorem.

The proposition that any rotation is a product of two line reflections corresponds to the proposition that any projectivity is a product of two involutions. The point reflection with O as center is commutative with all the other rotations about O and hence effects on C^2 an involution which (§ 79, Vol. I) belongs to all the projectivities effected on C^2 by the rotations of this group. This involution is harmonic (§ 78, Vol. I) to the involution effected on C^2 by any orthogonal line reflection whose axis contains O, and hence all the involutions of the latter sort form a pencil. Thus all the theorems of § 79, Vol. I, can be specialized so as to yield theorems about the group of rotations with O as center.

There are many other applications of the theorems in Chap. VI, Vol. I, to affine and Euclidean geometry (a few of them are indicated in the exercises below), but the main application which we are to consider at present is to the theory of order relations. Let us first recall some of the ordinal theorems which have already been estab-

ordered space is *hyperbolic, parabolic,* or *elliptic* according as it has two, one, or no double points.

With regard to involutions, we have already established the following propositions (§ 21): *If an involution preserves sense, each pair separates every other pair. If an involution alters sense, no pair separates any other pair. An involution which does not alter sense is elliptic;* that is to say, *the pairs of a hyperbolic involution do not separate each other. The double points of a hyperbolic involution separate every pair of the involution.*

DEFINITION. If A, B, C, D are four distinct points of a conic, the point O of intersection of the lines AB and CD is called an *interior* point in case the pairs AB and CD separate each other* and an *exterior* point in case these pairs do not separate each other. The set of all interior points is called the *interior* or *inside* of the conic, and the set of all exterior points is called the *exterior* or *outside* of the conic.

The pairs AB and CD are conjugate in the involution with O as center. Hence, if these two pairs separate each other, this involution preserves sense and is such that any two of its pairs separate each other. Hence any two lines through O which meet the conic meet it in pairs of points which separate each other. That is to say, the definition of an interior point is independent of the particular choice of the points A, B, C, D. A like argument applies in case O is exterior. In case the involution with O as center has double points, the lines joining O to these points are tangent to the conic. Hence the next to the last of the propositions about involutions stated above implies that there are no tangents through an *interior* point. These results may be stated as follows:

THEOREM 1. *The points coplanar with a conic fall into three mutually exclusive classes: the conic itself, its interior and its exterior. Each interior point is the center of an involution on the conic which preserves sense, and each exterior point of one which alters sense. All points of a tangent, except the point of contact, are exterior points of the conic.*

* Cf. § 20, particularly the footnote.

Now let O be any interior point. If O' is any point conjugate to O with regard to the conic, there exists (cf. fig. 59) a complete quadrangle $ABCD$ whose vertices are points on the conic such that AB and CD meet in O and AD and CB meet in O'. But by Theorem 7,

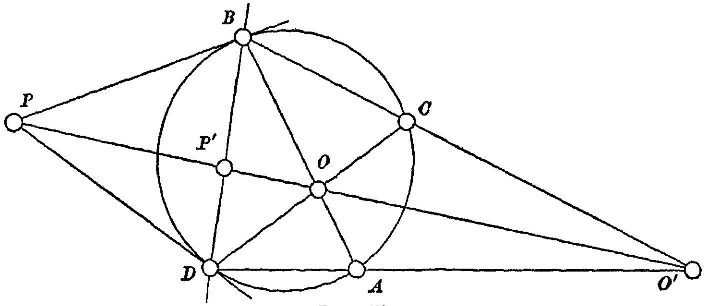


FIG. 59

Chap. II, if AB separates CD, then AD does not separate BC, and hence O' is an exterior point. Hence the polar line of any interior point consists entirely of exterior points. Hence

THEOREM 2. *All points conjugate to an interior point are exterior.*

Suppose, further, that the tangent to the conic at B meets the line OO' in a point P and the line BD meets OO' in a point P' (fig. 59). Then P and P' are conjugate points with regard to the conic. Moreover,

$$ABCD \;\overline{\wedge}\; OPO'P'.$$

Since A and C do not separate B and D, it follows that the pair OO' does not separate the pair PP'. That is,

THEOREM 3. *On a line containing an interior point of a conic the pairs of conjugate points with regard to the conic do not separate one another.*

By elementary propositions about poles and polar there follow at once:

COROLLARY 1. *The pole of a line which contains an interior point is an exterior point.*

COROLLARY 2. *The polar of an exterior point contains some in-*

In § 78, Vol. I, it was established that any projectivity is a product of two involutions one of which is hyperbolic. Since a hyperbolic involution is opposite, it follows that if the given projectivity is direct, it is a product of two opposite involutions; and if the given projectivity is opposite, it is a product of a direct and an opposite involution. But in the second case the direct involution is, by the argument just made, a product of two opposite involutions. Hence

THEOREM 4. *A direct projectivity is a product of two opposite involutions, and an opposite projectivity is a product of three opposite involutions. An opposite projectivity is also expressible as a product of a direct and an opposite involution.*

In the case of projectivities on a conic, the axis of the product of two involutions is the line joining their centers. Hence we have, as consequences of this theorem,

COROLLARY 1. *Any line in the plane of a conic contains points exterior to the conic.*

COROLLARY 2. *A projectivity whose center is an interior point, and whose axis therefore consists entirely of exterior points, is direct.*

In the fourth exercise, below, we need the following definition:

DEFINITION. The line perpendicular to a tangent to a conic and passing through its point of contact is called the *normal* to the conic at this point.

EXERCISES

1. What transformations of the Euclidean group effect projectivities on l_∞ to which the absolute involution belongs? How are these distinguished from the remaining similarity transformations by their relation to the circular points? What transformations of the Euclidean group are harmonic on l_∞ to the absolute involution?

2. Show that the measure of a line pair as defined in § 72 is the logarithm of the characteristic cross ratio of a certain projectivity on l_∞. Obtain an analogous formula for the measure of an angle in terms of the characteristic cross ratio of a projectivity on a circle.

3. Any noninvolutoric planar collineation which leaves invariant a conic and a line transforms the points of the line by a projectivity to which belongs

4. If P is any fixed point of a conic and RQ a variable point pair such that $\measuredangle RPQ$ is a right angle, the lines RQ meet in a fixed point on the normal at P.

5. The lines joining homologous points in a noninvolutoric projectivity on a conic are the tangents of a second conic.

6. If P is any fixed point of a conic and RQ a variable pair of points such that $\measuredangle RPQ$ has constant measure, the lines RQ are the tangents to a second conic.

7. If a projectivity Γ on a line is a product of an involution having double points, A_1 and B_1, followed by another involution, and if $\Gamma^{-1}(A_1) = A_0 \neq A_1$ and $\Gamma(A_1) = A_2$, then A_1 and B_1 are harmonically conjugate with regard to A_0 and A_2 whenever $A_0 \neq A_2$; and $B_1 = A_0$ whenever $A_0 = A_2$.

8. If A_1 and B_1 are a pair of an involution I which is left invariant by a projectivity Γ, and if $\Gamma^{-1}(A_1) = A_0 \neq A_1$ and $\Gamma(A_1) = A_2 \neq A_0$, then A_0 and A_2 are harmonically conjugate with regard to A_1 and B_1.

9. Let A and A' be any pair of an involution I. If $A \neq A'$, any projectivity Π which transforms I into itself and leaves A invariant is either the involution, with A and A' as double points, or the identity.

10. Generalize § 80, Vol. I, so as to apply to the group of translations and the equiaffine group, using the fact that the transformations in each of these groups are products of pairs of involutoric projectivities.

75. Interior and exterior of a conic.

THEOREM 5. *Any two points of a conic are the ends of two linear segments one consisting entirely of interior points and the other entirely of exterior points.*

Proof. Let the given points be denoted by A and B, let C and D be any two other points of the conic which separate A and B, and let σ and $\bar{\sigma}$ represent the segments $\overline{ACB}$ and $\overline{ADB}$ on the conic. By the definition of the order relations on the conic, the lines joining C to the points of $\bar{\sigma}$ meet the line AB in the points of a segment $\bar{\sigma}'$ whose ends are A and B, and these points satisfy the definition of interior points. In like manner the lines joining C to points of σ meet the line AB in a segment σ' which is complementary to $\bar{\sigma}'$ and consists entirely of exterior points.

In a real plane the following theorem is a consequence of what we have just proved, but in order to have the result for any ordered plane we give a proof which is entirely general.

THEOREM 6. *Any two interior points of a conic are the ends of a*

Proof (fig. 60). Let A and C be two interior points. Let A_1 be any nt of the conic not on the line AC. The lines A_1C and A_1A are not gent to the conic, since (Theorem 1) the involutions at A and C are h elliptic. Let A_0 and B_2 respectively be the points, distinct from in which the lines A_1A and A_1C meet the conic. The two segments the conic whose ends are A_0 and B_2 are projected by the lines ough A_1 into the two segments of the line AC which have A and s their ends. We shall prove that the segment σ of the line AC ich is the projection of the segment complementary to $\overline{A_0A_1B_2}$ con-s entirely of interior points.

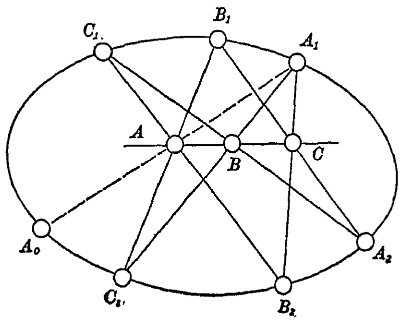


FIG. 60

Let B be any point of σ. The e A_1B then meets the conic in oint C_2 which is separated from by A_0 and B_2. Let B_2A meet conic in C_1, let C_1B meet it in and let A_2C meet it in B_1, so t $A_1B_2C_1A_2B_1C_2$ form a Pascal xagon whose pairs of opposite es meet in A, B, C. Since A is interior point, we have the er $\{C_2A_0B_1A_1\}$. Since B was chosen so that C_2 and A_1 are separated by and A_0, we have $\{B_2C_2A_0A_1\}$. From these there follows $\{B_2C_2A_0B_1A_1\}$. nsforming this by the involution at A we have $\{C_1B_1A_1C_2A_0\}$. Hence have $\{B_2C_2A_0C_1B_1A_1\}$. Since the involution with center at C is ptic, we have $\{B_2B_1A_1A_2\}$. Hence we have $\{B_2C_2A_0C_1B_1A_1A_2\}$. Hence and A_1 separate A_2 and C_1, and hence B is interior to the conic.

THEOREM 7. *Any two exterior points are ends of a segment consist- entirely of exterior points.*

Proof. Let the two exterior points be E_1 and E_2. If the line E_1E_2 tangent, all points on it except the points of contact are exterior, ce each of these points is the center of a hyperbolic involution the conic. In this case the theorem is obvious. If the line E_1E_2 eets the conic in two points, the theorem reduces to Theorem 5. the line E_1E_2 does not meet the conic, and both the segments with

The theorems above are connected with the following algebraic considerations: Any involution can be written in the form

(1) $$x' = \frac{ax + b}{cx - a}.$$

If we regard a, b, c as a set of homogeneous coördinates in a projective plane, then for every involution (1) there is one and only one point (a, b, c); and inversely for every point (a, b, c) there is a unique involution (1), provided that the point does not satisfy the condition

(2) $$a^2 + bc = 0.$$

By § 18 the projectivities (1) for which

(3) $$a^2 + bc > 0$$

are opposite, and those for which

(4) $$a^2 + bc < 0$$

are direct.

The equation (2) *represents a conic section of which the points satisfying* (3) *are the exterior and those satisfying* (4) *are the interior.* This may be proved as follows:

The conic is given by the parametric representation (§ 82, Vol. I)

$$a : b : c = x : x^2 : -1,$$

and any involution on the conic is given by the transformation (1) of the parameter x. The center of the involution is the point of intersection of the lines containing pairs of the involution. The point $(0, 0, 1)$ of the conic is given by the value 0 of the parameter x and thus is transformed to the point given by the value $x = -b/a$, namely, the point $(-ab, b^2, -a^2)$. The point $(0, 1, 0)$ of the conic is given by $x = \infty$ and thus is transformed to the point given by $x = a/c$, namely, the point $(ac, a^2, -c^2)$. The point of intersection of the lines joining $(0, 0, 1)$ to $(-ab, b^2, -a^2)$ and $(0, 1, 0)$ to $(ac, a^2, -c^2)$ is manifestly $(-a, b, c)$. Hence $(-a, b, c)$ is the center of the involution (1), and therefore is interior to the conic if (4) is satisfied and the involution direct, and exterior to the conic if (3) is satisfied and the involution opposite.

EXERCISES

1. Parabolic projectivities are direct.

2. Two of the three vertices of any self-polar triangle of a conic are exterior points.

3. The center of a hyperbola is an exterior point.

4. The center of a circle is an interior point.

5. In a Euclidean plane all points interior to a circle and all points on it

6. If a segment A_1B_1 is contained in a segment A_2B_2, the circle the ends whose diameter are A_1 and B_1 is composed of points interior to the circle ends of whose diameter are A_2 and B_2.

7. In a Euclidean plane all points interior to an ellipse lie entirely on side of any line consisting entirely of exterior points.

8. Any two pairs of conjugate diameters of an ellipse separate each other o pairs of conjugate diameters of a hyperbola never separate each other.

9. If O is the center of a conic K^2, the polar reciprocal of a conic C^2 h respect to K^2 will be an ellipse, parabola, or hyperbola according as O nterior to, on, or exterior to C^2.

10. Consider a conic C^2 in a planar net of rationality satisfying Assump- 1 II. The points of the net exterior to the conic fall into two classes $[E]$ l $[F]$ such that two tangents to the conic can be drawn from any point E l no tangent can be drawn to the conic from any point F. On any line in ich one E is conjugate to an F with regard to C^2, every E is conjugate to F. On any line in which one E is conjugate to an E, every E is conjugate an E and every F to an F. The interior points fall into two classes $[I]$ l $[J]$ such that the pairs of conjugate lines on a point I either both meet or both do not meet C^2, whereas one member of any pair of conjugate lines a point J meets C^2 and the other member does not meet C^2.

11. Let the equation of a conic be $f(x_0, x_1, x_2) = 0$ and let the determinant the coefficients of $f(x_0, x_1, x_2)$ be

$$A = \begin{vmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{vmatrix} \neq 0, \qquad a_{ij} = a_{ji}.$$

point (x_0', x_1', x_2') is interior or exterior according as $A \cdot f(x_0', x_1', x_2')$ is ater or less than zero.

76. Double points of projectivities. The preceding theorems hold any ordered space. On specializing to a real space we have the ditional theorem that a projectivity which alters sense has two uble points (§ 4). In the case of involutions this result com- ned with the theorem that a hyperbolic involution is always posite gives

THEOREM 8. *The pairs of an elliptic involution always separate e another, and the pairs of a hyperbolic involution never separate e another.*

The last half of this theorem, combined with Theorem 3, gives e condition for the intersection of a line with a conic, a condition

conic in the double points of this involution.

By Cor. 2, Theorem 3, the polar of an exterior point is a line through an interior point. The lines joining the exterior point to the points of intersection of its polar with the conic are tangents. Hence

COROLLARY 1. *Through any exterior point there pass two tangents to a conic.*

COROLLARY 2. *Two involutions, one at least of which is elliptic, have one and only one common pair.*

Proof. The center of an elliptic involution represented on a conic is an interior point. The line joining this point to the center of any other involution meets the conic in two points which are pairs of both involutions. Since any pair of an involution is collinear with the center, the two points so constructed are the only pair common to the two involutions.

A special case of this corollary may be stated in the following form:

COROLLARY 3. *In a given one-dimensional form there is one and only one pair of elements which are conjugate with respect to a given elliptic involution and harmonically separated by a given pair of elements.*

Since a hyperbolic involution is determined by its double points, it is evident that any two hyperbolic involutions are equivalent under the group of all projectivities of a one-dimensional form. The corresponding theorem for elliptic involutions is best seen by representing the involutions on a conic. The two centers I_1, I_2 are interior points, and the line joining them meets the conic in two points C_1, C_2 which do not separate them (Theorem 5). Let O_1 and O_2 be the double points (Theorem 8) of the involution in which I_1I_2 and C_1C_2 are pairs. An involution with either of the points O_1 or O_2 as center will evidently transform the one with I_1 as center into the one with I_2 as center. Hence

COROLLARY 4. *Any two elliptic involutions in the same real one-dimensional form are conjugate under the projective group of that form.*

EXERCISES

1. All involutions which are harmonic to (i.e. commutative with and distinct from) an elliptic involution are hyperbolic.

2. If two points A, B of a line separate each point $P (P \neq A,\ P \neq B)$ of the line from its conjugate point in a given elliptic involution, A and B are conjugate in this involution.

3. A hyperbolic projectivity is opposite or direct according as a pair of homologous points does or does not separate the double points.

4. Elliptic projectivities are direct.

5. The center of an ellipse is an interior point.

6. The involution determined on the line at infinity of a Euclidean plane by an ellipse is elliptic, by a hyperbola, hyperbolic.

7. Any two ellipses are conjugate under the affine group.*

8. An involution in a flat pencil is either such that every pair of conjugate lines is orthogonal or there is one and only one orthogonal pair of conjugate lines.

9. A conic having two pairs of perpendicular conjugate diameters is a circle.

10. If A_1 and A_2 are the real limiting points of a pencil of circles, each circle of the pencil either contains A_1 and is on the opposite side of the radical axis from A_2, or contains A_2 and is on the opposite side of the radical axis from A_1.

11. Of two circles of a pencil, both containing the same limiting point, one is entirely interior to the other.

12. For any angle, $\measuredangle ABC$, there is one and only one pair l, l' of orthogonal lines through B which separate the lines BA and BC harmonically. One line, l, of the pair contains points P interior to $\measuredangle ABC$, and $\measuredangle ABP$ is congruent to $\measuredangle PBC$. The line l is called the *interior bisector*, and the line l' the *exterior bisector*, of the angle $\measuredangle ABC$.

13. The asymptotes of an equilateral hyperbola bisect any pair of conjugate diameters.

14. The bisectors of the angles of a triangle ABC meet in four points, one in each of the four regions determined by ABC according to § 26. These four points are the centers of four circles inscribed in ABC and are the vertices of a complete quadrilateral of which ABC is the diagonal triangle. The mid-point of the pair BC is the mid-point of the points of contact of either pair of inscribed circles whose centers are collinear with A.

15. Let V and V' be the vanishing points (§ 43) of a projectivity on a line, the notation being so assigned that the point at infinity is transformed to V'. There exist two points A, B which are transformed to two points A', B' such that

$$AV = VB = A'V' = V'B'.$$

77. Ruler-and-compass constructions. The discussion in Chap. IX, Vol. I, reduces any quadratic problem to the problem of finding the points of intersection of an arbitrary line with a fixed conic. According to Theorems 5 and 9 the necessary and sufficient condition that a line coplanar with a conic meet it in two points is that the line pass through an interior point of the conic. Hence this condition will serve to determine the solvability of any problem of the second degree in a real space. Thus the discussion of linear and quadratic constructions, under the projective meaning of these terms, may be regarded as complete.

When we adopt the Euclidean point of view, the fixed conic may be taken as a circle; and therefore every problem of the second degree is reduced to the problem of determining the points of intersection of an arbitrary line with a fixed circle (cf. § 86, Vol. I).

The constructions of elementary Euclidean geometry which are known as ruler-and-compass constructions involve the determination of the points of intersection (whenever existent) of two arbitrary lines, or of an arbitrary line with an arbitrary circle, or of two arbitrary circles. The last of these problems has been shown in § 65 to be reducible to the first and second. Hence any ruler-and-compass construction may be reduced to the problem of finding the intersection of an arbitrary line with a fixed circle.

On account of the special character of the line at infinity, there is not a perfect correspondence between the linear constructions of projective geometry and the Euclidean constructions by means of a ruler. The operations involved in the linear constructions of projective geometry are

(*a*) to join two points by a (projective) line;
(*b*) to take the point of intersection of any two lines.

These are evidently equivalent to the following Euclidean operations:

(1) to join two ordinary points by a line;
(2) to take the point of intersection of two nonparallel lines;
(3′) to draw a line through a given point parallel to a given line.

The first of these operations corresponds to the proposition that

may be thought of as carried out with a straightedge or ruler whose length is not limited.

The operation (3′) can be effected by means of (1) and (2), together with the following operation:

(3) to find on any ray through a point A, a point C such that the point pair AC is congruent to a preassigned point pair AB.*

For let A be the given point and let BC be the given line. Let O be a point on the line AB in the order $\{ABO\}$ such that BA is congruent to BO. Let $\bar{A}$ be the point of the line OC in the order $OC\bar{A}$ such that CO is congruent to $C\bar{A}$. Then $A\bar{A}$ is evidently parallel to BC.

Thus (1), (2), and (3) serve as a basis for all linear operations in the projective sense. They obviously yield also a certain class of quadratic constructions; but they do not suffice for all quadratic constructions. The latter may be provided for, as explained above, by adjoining the operation of taking the point of intersection with a fixed circle of an arbitrary line through an arbitrary interior point.

For the proof that (3′) is not a consequence of (1) and (2), and that (1), (2), (3) do not provide for all quadratic constructions, the reader is referred to Hilbert, Grundlagen der Geometrie, Chap. VII (4th edition, 1913).

EXERCISES

1. Given three collinear points A, B, C such that AB is congruent to BC, show how to construct a parallel to the line AB through an arbitrary point P by means of the operations (1) and (2) alone.

2. Given two parallel lines, show how to find the mid-point of any pair of points on either of the lines by means of (1) and (2) alone.

3. Given a parallelogram and a point P and a line l in its plane. Through P draw a line parallel to l, making use of the ruler only.

* It is important to notice that the pairs AB and AC have the point A in common. Thus (3) provides merely for drawing a circle through a given point and with a given other point as center. The drawing instrument to which this corresponds is a pair of compasses which snaps together when lifted from the paper, so that it cannot be used to transfer a point pair AB to a point pair $A'B'$ unless $A = A'$. This will be understood by anyone reading the second proposition in Euclid's Elements, which shows how to lay off a point pair congruent to a given point pair on a given ray. The operation (3) may be replaced by the operation of finding on any ray AB a point C such that the point pair AC is congruent to a fixed point pair OP. The instrument for this operation may be thought of as a measuring rod of fixed length (say unit length) without subdivisions. (Cf. the reference

4. Given a point pair AC and its mid-point B, using the ruler alone, construct the point pair AD such that

$$\frac{AC}{AD} = n.$$

5. Given four collinear points A, A', B, B', construct the fixed point of the parabolic projectivity carrying A to A' and B to B'.

6. Given a projectivity on a line, find a pair of corresponding points A and A' such that a given point M is the mid-point of the segment AA'.

7. Inscribe in a given triangle a rectangle of given area.

8. Given four tangents of a parabola, construct a tangent parallel to a given line.

9. Given three points of a hyperbola and a line parallel to each asymptote, find the point of intersection of the hyperbola with a line parallel to one of the asymptotes.

10. Construct by ruler and compass any number of tangents to a conic given by five of its points; also any number of points of a conic given by five of its tangents.

11. Construct any number of points of a parabola through four given points.

12. Construct any number of points of a parabola touching three given lines and passing through a given point.

13. Through a given point construct an orthogonal pair of lines conjugate with regard to a conic. (If the point is exterior to the conic, these lines are the bisectors of the angles formed by the tangents to the conic from this point.)

78. Conjugate imaginary elements. It has been shown in § 6 that a real projective space S can be regarded as immersed in a complex projective space S′ in such a way that every line of S is a subset of a unique line of S′. Certain additional definitions and conventions have been introduced in § 70. But in both these places little use was made of the properties of imaginary elements beyond their existence and the fact that S′ satisfies Assumptions A, E, P. We shall now prove some of the most elementary theorems about the relation between elements of S and S′.

DEFINITION. Two imaginary points, lines, or planes are said to be *conjugate relative to a real one-dimensional form* of the first or second degree if and only if they are the double elements of an involution in the real form.

As an example consider a real conic C^2 and a line l exterior to it. The conic and the line have in common the double points of an ellip-

the involution on C^2 whose axis is l. Hence the points common to C^2 and l are conjugate imaginaries both with respect to C^2 and to l. Since any one-dimensional form of the first or second degree whose elements are points is a line or a point conic, and since the double points of any involution on a conic are the intersections of the axis of the involution with the conic, we have

THEOREM 10. *Any two conjugate imaginary points are on a real line.*

By duality we have that any two conjugate imaginary planes are on a real line.

Two conjugate imaginary lines are by definition on a real point, line conic, cone of lines, or regulus. If they are on a real line conic, the plane dual of the argument above shows that they are on a real point. By dualizing in space we obtain the same result for conjugate imaginary lines of a cone of lines. Hence we have

THEOREM 11. *Any two conjugate imaginary coplanar lines are on a real point and any two conjugate imaginary concurrent lines are on a real plane.*

Conjugate imaginary lines on a regulus will be considered in a later chapter.

THEOREM 12. *The lines joining a real point to two conjugate imaginary points not collinear with it are conjugate imaginary lines.*

Proof. The conjugate imaginary points are double points of an elliptic involution on a real line. From any point not on this line this involution is projected into an involution of lines whose double lines are the projections of the given points.

THEOREM 13. *If A_1A_2 and B_1B_2 are two pairs of conjugate imaginary points on different lines, the lines A_1B_1 and A_2B_2 meet in a real point and are conjugate imaginary lines.*

Proof. By hypothesis the lines A_1A_2 and B_1B_2 are real and hence they meet in a real point C. Let B be the conjugate of C in the elliptic

the two real points which are paired in this involution and separate A and C harmonically. Since any two harmonic sets are projective,

$$CBPQ \overline{\wedge} CARS \quad \text{and} \quad CBPQ \overline{\wedge} CASR.$$

The centers of these two perspectivities are two real points C_1 and C_2, and since each perspectivity transforms two pairs of the elliptic involution on the line A_1A_2 into two pairs of the elliptic involution on the line B_1B_2, it transforms A_1 and A_2 to B_1 and B_2. Hence one of the points C_1 and C_2 is the intersection of the lines A_1B_1 and A_2B_2 and the other that of the lines A_1B_2 and A_2B_1. By Theorem 12 each of these pairs of lines is a pair of conjugate imaginaries.

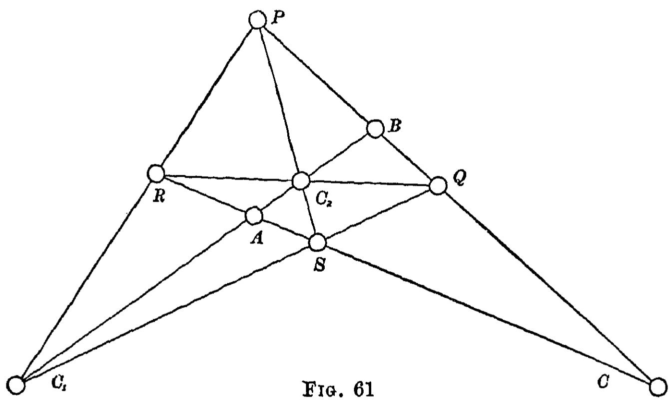


FIG. 61

The complete quadrilateral whose pairs of opposite vertices are A_1A_2, B_1B_2, and C_1C_2 is analogous to the quadrilateral considered in § 71 whose vertices were I_1I_2 and the limiting points of two orthogonal pencils of circles (cf. fig. 57). With regard to the existence of such quadrilaterals we have

THEOREM 14. *Let A_1A_2, B_1B_2, C_1C_2 be the pairs of opposite vertices of a complete quadrilateral. If A_1A_2 and B_1B_2 are pairs of conjugate imaginary points, then C_1 and C_2 are real and the diagonal triangle of the complete quadrilateral is real. If A_1 and A_2 are real and B_1 and B_2 are conjugate imaginaries, then C_1 and C_2 are conjugate imaginaries and the diagonal triangle is real.*

Proof. In the first case C_1 and C_2 are determined as in the proof of the last theorem and hence are real. The diagonal triangle has for

In the second case let a be the line through A_2 which is harmon-
ally conjugate to A_2A_1 with respect to the pair of lines A_2B_1 and
B_2. Since the latter two lines are conjugate imaginaries and A_2A_1
real, a is real. The harmonic homology with A_1 as center and a as
is transforms B_1 and B_2 to C_1 and C_2. Hence C_1 and C_2 are conjugate
aginaries and the line C_1C_2 is real.

Relatively to a real frame of reference a real involution is repre-
nted by a bilinear equation with real coefficients (§ 58, Vol. I), and
double points appear as the roots of a quadratic equation with real
efficients. Hence the coördinates of a pair of conjugate imaginary
ints are expressible in the form

$$(x_0 + iy_0,\ x_1 + iy_1,\ x_2 + iy_2,\ x_3 + iy_3)$$

d

$$(x_0 - iy_0,\ x_1 - iy_1,\ x_2 - iy_2,\ x_3 - iy_3),$$

ere $x_0, x_1, x_2, x_3, y_0, y_1, y_2, y_3$ are real. Like remarks can be made
th regard to the coördinates of a plane or a line, and Theorems 10–14
n easily be proved analytically on this basis. The following theorem
pears to be easier to prove analytically than synthetically:

THEOREM 15. *A complex line on a real plane contains at least one
al point.*

Proof. Let the equation of the line be

$$u_0x_0 + u_1x_1 + u_2x_2 = 0.$$

is may be expressed in the form

$$(u_0' + iu_0'')\, x_0 + (u_1' + iu_1'')\, x_1 + (u_2' + iu_2'')\, x_2 = 0,$$

here u_0', u_0'', etc. are real. This equation is equivalent, if x_0, x_1, x_2
e required to be real, to

$$u_0'x_0 + u_1'x_1 + u_2'x_2 = 0,$$
$$u_0''x_0 + u_1''x_1 + u_2''x_2 = 0,$$

o equations which are satisfied by at least one real point.

EXERCISES

1. A conic section through three real and two conjugate imaginary points
real.

2. A pair of conjugate imaginary points cannot be harmonically conjugate
ith regard to another pair of conjugate imaginary points.

3. An imaginary point is on one and only one real line and has one and

79. Projective, affine, and Euclidean classification of conics. Let us regard a real plane π as immersed in a complex plane π', and consider all conics in π' with respect to which the polar of a real point is always a real line.*

Throughout the rest of this chapter the word "conic" shall be used in this sense. The involution of conjugate points with regard to such a conic is one in which real points are paired with real points. Hence, if a conic contains one real point, every real nontangent line through this point contains another point of the conic, and the conic is real. The conics under consideration therefore fall into two classes, the real conics† and those containing no real point.

By § 76, Vol. I, any two real conics are equivalent under the group of projective collineations. The same proposition holds also for any two conics of the other class, as we shall now prove. Let two such conics be denoted by C_1^2 and C_2^2. On an arbitrary real line l they each determine an elliptic involution of conjugate points. By Cor. 4, Theorem 9, there is a projectivity of the line l carrying the involution determined by C_2^2 into that determined by C_1^2. Any projectivity of the real plane which effects this transformation on l will carry C_2^2 into a conic C_3^2 which has the two conjugate imaginary points A_1, A_2 on l in common with C_1^2. A collineation leaving l invariant will now carry the pole of l with regard to C_3^2 to the pole of l with regard to C_1^2; and therefore carries C_3^2 to a conic C_4^2 which has A_1, A_2 and the tangents at these points in common with C_1^2. Let L be the pole of l with regard to C_1^2 and L_1 be any real point of l. By Cor. 3, Theorem 9, there is a pair of points MM_1 which are conjugate with respect to C_1^2 and harmonically separate L and L_1 and also a pair $M'M_1'$ conjugate with respect to C_4^2 and harmonically separating L and L_1. The homology with l as axis, L as center, and carrying M' to M carries C_4^2 to C_1^2. Hence we have

THEOREM 16. *Any two real conics or any two imaginary conics with real polar systems are conjugate under the group of real projective collineations.*

* In § 85 this condition is seen to be equivalent to the condition that the equation of the conic relative to a frame of reference in π shall be expressible with

If the line l be taken as the line at infinity of a Euclidean plane the argument above shows that any two imaginary conics are also conjugate under the affine group. Since these conics do not meet any real line in real points, they are analogous to ellipses no matter how the line at infinity is chosen. Hence we make the definition:

DEFINITION. An imaginary conic with a real polar system is called an *imaginary ellipse.*

The results just established, together with those stated in Ex. 7, § 76, and Exs. 14 and 15, § 37, may be summarized as follows:

THEOREM 17. *Under the affine group the conics with real polar systems fall into four classes, parabolas, hyperbolas, real ellipses, imaginary ellipses. Any two conics of the same class are equivalent.*

Under the Euclidean group conics must be characterized by their relations to the circular points I_1, I_2. Since a real conic which does not meet l_∞ in real points meets it in conjugate imaginary points, any real conic through I_1 also contains I_2 and is therefore a circle. For the same reason the imaginary conic determined by an elliptic polar system must contain I_2 if it contains I_1.

DEFINITION. An imaginary ellipse with respect to which the pairs of conjugate points on l_∞ are pairs of the absolute involution is called an *imaginary circle.*

THEOREM 18. *Any two real circles or any two imaginary circles are similar.*

Proof. Let the centers, necessarily real, of two circles C^2 and K^2 be O_1 and O_2 respectively. The center O_1 may be transformed to O_2 by a translation T_1. This carries C^2 to a circle C_1^2. Any real line l through O_2 meets C_1^2 in two points C_1 and C_2 and K^2 in two points K_1 and K_2. Since each of these pairs is harmonically conjugate with respect to O_2 and the point at infinity O_∞ of l, the homology T_2 with O_2 as center and l_∞ as axis which carries C_1 to K_1 also carries C_2 to K_2. This homology evidently carries all real points to real points if C_1, C_2, K_1, K_2 are real. If C_1C_2 and K_1K_2 are pairs of conjugate imaginary points, consider (§ 77) the real pair of points PP' harmonically conjugate with regard to C_1C_2 and OO_∞ and the real pair QQ' harmonically conjugate with regard to K_1K_2 and OO_∞. The homology T_2 must carry P and P' to Q and Q' and therefore carries all real points to real points in

Now the conic C_1^2 is fully determined by its points I_1, I_2, C_1, C_2 and its center O_2 and K^2 is fully determined by I_1, I_2, K_1, K_2 and O_2. Hence T_2 carries C_1^2 to K^2. The product $\mathrm{T}_2\mathrm{T}_1$ carries C^2 to K^2.

THEOREM 19. *Any two parabolas are similar.*

Proof. Let C^2 and K^2 be two parabolas and let C_∞ and K_∞ be their points of contact with l_∞. Let T_1 be any rotation carrying C_∞ to K_∞ and let $\mathrm{T}_1(C^2) = C_1^2$. Let $\overline{K}_\infty$ be the conjugate of K_∞ in the absolute involution and let c be the ordinary line through $\overline{K}_\infty$ tangent to C_1^2 and C its point of contact; also let k be the ordinary line through $\overline{K}_\infty$ tangent to K^2, and K its point of contact. The translation T_2

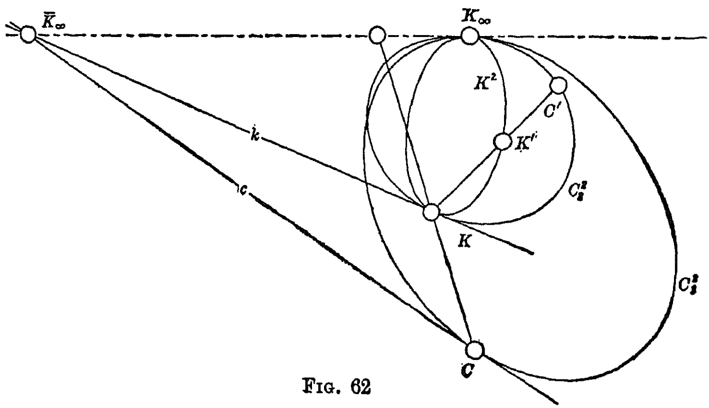


FIG. 62

carrying C to K carries c to k and C_1^2 to a conic C_2^2 touching l_∞ at K_∞. Any line l through K, not containing K_∞ or $\overline{K}_\infty$, meets C_2^2 in a point C' and K^2 in a point K'. The homology T_3 with K as center, l_∞ as axis, and carrying C' to K' carries C_2^2 to K^2. The product $\mathrm{T}_3\mathrm{T}_2\mathrm{T}_1$ is a similarity transformation carrying C^2 to K^2.

No theorem analogous to the last two holds for ellipses and hyperbolas. Suppose an ellipse or a hyperbola C^2 meets l_∞ in C_1 and C_2 and another ellipse or hyperbola K^2 meets it in K_1 and K_2. In case a similarity transformation carries C_1 and C_2 into K_1 and K_2,

(5) $$\mathrm{R}(I_1I_2,\, C_1C_2) = \mathrm{R}(I_1I_2,\, K_1K_2).$$

Conversely, if C^2 and K^2 satisfy the condition (5) there evidently exists a rotation carrying C_1 and C_2 to K_1 and K_2. This rotation carries C^2 to a conic C_1^2 which passes through K_1 and K_2. By an argument

analogous to the proof of Theorem 18 it can be shown that if C_1^2 and K^2 are both real ellipses, or both imaginary ellipses, or both hyperbolas, there is a similarity transformation carrying C_1^2 to K^2. Hence

THEOREM 20. *Two real ellipses or two imaginary ellipses or two hyperbolas which meet l_∞ in pairs of points C_1C_2 and K_1K_2 are similar if and only if* $\mathrm{R}(I_1I_2, C_1C_2) = \mathrm{R}(I_1I_2, K_1K_2)$.

EXERCISE

A hyperbola for which $\mathrm{R}(I_1I_2, K_1K_2) = -1$ is rectangular (Ex. 3, § 73).

80. Foci of the ellipse and hyperbola. Let C^2 be any hyperbola or real or imaginary ellipse, and let l_1, l_2 be the tangents to C^2 through I_1 and l_3, l_4 the tangents to C^2 through I_2. The circular points I_1, I_2 are one pair of opposite vertices of the complete quadrilateral $l_1l_2l_3l_4$. Let the other two pairs of opposite vertices be F_1F_2 and $F_1'F_2'$ respectively (fig. 63), let a be the line F_1F_2, b the line $F_1'F_2'$, and O the point of intersection of a and b. Also let A_∞ and B_∞ be the points at infinity of the lines a and b respectively. The triangle $OA_\infty B_\infty$ is self-polar with respect to C^2. Hence O is the center of C^2 and is therefore real.

Let X be any real point not on l_1, l_2, l_3, l_4 or C^2. By the dual of the Desargues theorem on conics (§ 46, Vol. I) the tangents to C^2 through X are paired in the same involution with XI_1, XI_2 and XF_1, XF_2 and XF_1', XF_2'. The double lines x_1, x_2 of this involution are harmonically conjugate with regard to XI_1, XI_2 and to the tangents to C^2. Hence they are paired both in the involution of orthogonal lines at X and the involution of lines conjugate with respect to C^2 at X. Hence by Cor. 2, Theorem 9, x_1 and x_2 are real, and are the unique pair of orthogonal lines on X which are conjugate with regard to C^2.

In particular, if $X = O$ it follows that a and b are real and are the only pair of orthogonal and conjugate diameters of C^2. Hence A_∞ and B_∞ are also real. If X is not on a, b, or l_∞, the lines x_1 and x_2 meet a in a pair of real points X_1, X_2 distinct from A_∞ and O. Since F_1 and F_2 are harmonically conjugate with respect to the real pairs X_1X_2 and $A_\infty O$, they are either real or conjugate imaginaries. But since I_1 and I_2 are conjugate imaginaries, by Theorem 14 if one of the pairs F_1F_2 and $F_1'F_2'$ is a pair of real points, the other is a pair of conjugate imaginaries, and conversely. Hence the notation may be so assigned that F_1 and F_2 are real and F_1' and F_2' are conjugate imaginaries.

Let A_1 and A_2 be the points in which a meets C^2 and B_1 and B_2 the points in which b meets C^2. By construction neither of the lines a and b can be tangent to C^2 so that each of the pairs A_1A_2 and B_1B_2 is either real or a pair of conjugate imaginaries.

In case C^2 is an imaginary ellipse, both A_1A_2 and B_1B_2 are necessarily pairs of conjugate imaginaries. In case C^2 is a real ellipse, the line l_∞ does not meet it in any real point, and hence O, the pole of l_∞, is an interior point. Hence both a and b meet C^2 in real points. Hence if C^2 is an ellipse, A_1, A_2, B_1, B_2 are all real. Whether C^2 is an ellipse or a hyperbola, the tangents to C^2 from F_1 are conjugate imaginary lines since they join the real point F_1 to the conjugate

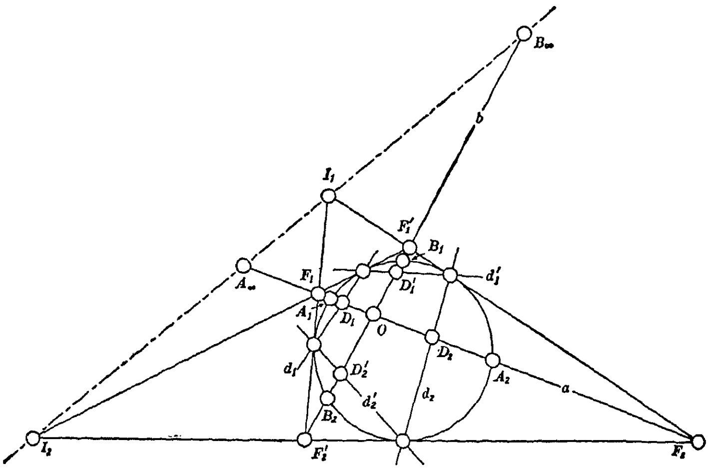


FIG. 63

imaginary points I_1 and I_2. Hence F_1 is interior to C^2, as is also F_2 by a like argument. Hence the line F_1F_2 meets C^2 in real points. Hence if C^2 is a hyperbola, A_1 and A_2 are real. But if C^2 is a hyperbola, O is an exterior point, and hence A_∞, which is harmonically separated from O by A_1 and A_2, must be an interior point. Hence b, the pole of A_∞, does not meet C^2 in real points, and consequently B_1 and B_2 are conjugate imaginaries.

Let the polars of F_1, F_2, F_1', F_2' relative to C^2 be denoted by d_1, d_2, d_1', d_2' respectively. Then d_1 and d_2 being the polars of real points are real; and since their point of intersection is polar to a, it is B_∞, and hence they are parallel to b. In like manner d_1' and d_2' pass through A_∞ and are conjugate imaginaries.

e conic C^2, a being called the *major*, or *principal*, axis and b the
nor, or *secondary*, axis. Each of the points F_1, F_2, F_1', F_2' is called a
cus, and each of the points A_1, A_2, B_1, B_2 a *vertex*, of the conic C^2.
ch of the lines d_1, d_2, d_1', d_2' is called a *directrix* of C^2.

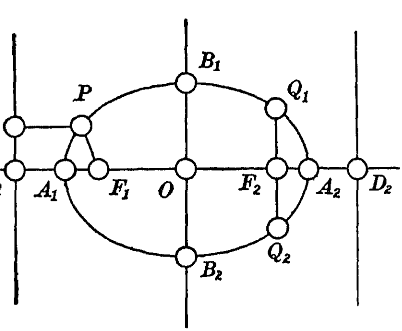


FIG. 64

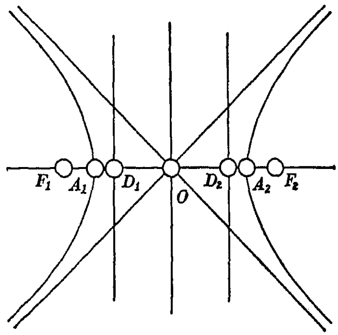


FIG. 65

In the course of the discussion of the complete quadrilateral $l_1l_2l_3l_4$ have established the following propositions:

THEOREM 21. *If C^2 is a hyperbola or a real or imaginary ellipse ich is not a circle, its axes are the unique pair of conjugate diam- rs which are mutually perpendicular. Two of the foci and two of the rectrices are real. The real foci lie on the major axis and the real rectrices are perpendicular to it. The other two foci are conjugate aginaries and lie on the minor axis. If C^2 is real, the real foci are terior points and the real directrices are exterior lines. If C^2 is a al ellipse, all four of the vertices are real; if C^2 is a hyperbola, the o vertices on the major axis are real and those on the minor axis e conjugate imaginaries.*

The two tangents to C^2 through F_1 pass also through I_1 and I_2. irs of conjugate lines at F_1 are separated harmonically by these o tangents and hence meet l_∞ in pairs of the involution whose uble points are I_1 and I_2. If we limit attention to real elements, is may be expressed by saying that the pairs of conjugate lines with spect to C^2 which pass through a focus are orthogonal. Conversely, the pairs of orthogonal lines at any point P are conjugate with spect to C^2, the double lines of the involution of orthogonal lines at

P would have to coincide with the double lines of the involution of conjugate lines, and hence P would be a focus. Hence

THEOREM 22. *The real foci of a hyperbola or a real or imaginary ellipse are the unique pair of real points at which all pairs of conjugate lines are orthogonal.*

The set of all conics tangent to the four minimal lines l_1, l_2, l_3, l_4 form a range (§ 47, Vol. I). Hence the pairs of tangents to these conics through any point P not on the sides of the diagonal triangle $OA_\infty B_\infty$ form an involution among the pairs of which are the pairs of lines PI_1, PI_2; PF_1, PF_2; and PF_1', PF_2'. Now if P is on C^2, there is only one tangent to C^2 at P, and this tangent is therefore a double line of the involution. This and the other double line have to be harmonically conjugate with respect to PI_1 and PI_2; that is, if C^2 and P are real, the two double lines have to be orthogonal. These double lines must be harmonically conjugate also with respect to PF_1 and PF_2. Thus we have a result which may be expressed as follows (cf. Ex. 12, § 76):

THEOREM 23. *The tangent and the normal to a real ellipse or hyperbola at any real point are the bisectors of the pair of lines joining this point to the real foci.*

In the proof of this proposition we have excepted the vertices of the conic, but the validity of the proposition for these points is self-evident. Another proposition which follows directly from the discussion above is the following, in which we make use of the fact that the pair of real foci determines the pair of imaginary ones, and vice versa.

THEOREM 24. DEFINITION. *The system of all conics having two real or two imaginary foci in common is a range of conics of Type I. The two conics of the set which pass through any real point have orthogonal tangents at this point. Such a range of conics is called a system of* confocal conics *or of* confocals.

The construction for the foci which has been considered in this section, when applied to a circle, reduces to a very simple one. The tangents to the circle at I_1 and I_2 meet in the center of the circle. The center of the circle is therefore sometimes referred to as the *focus* and the line at infinity as the *directrix*.

In the rest of the chapter the foci, center, directrices, and axes of an ellipse or a hyperbola will be denoted by the same letters as in this section. The notation has been assigned so that for an ellipse the points are in the order

$$\{D_1A_1F_1OF_2A_2D_2\},$$

and for the hyperbola in the order

$$\{F_1A_1D_1OD_2A_2F_2\},$$

where D_1 and D_2 denote the points of intersection of the principal axis with the directrices d_1 and d_2 respectively.

81. Focus and axis of a parabola. Let C^2 ...
it is tangent to l_∞, there are two ord...
and I_2 respectively; let these be
Let their point of intersection be de
with C^2 by L_1 and L_2 respectively, a
the point of contact of C^2 with l_∞ be
a, and the point, other than A_∞, in w

DEFINITION. The point F is called the *focus*, the line d the *directrix*, the line a the *axis*, the point A the *vertex*, of the parabola C^2.

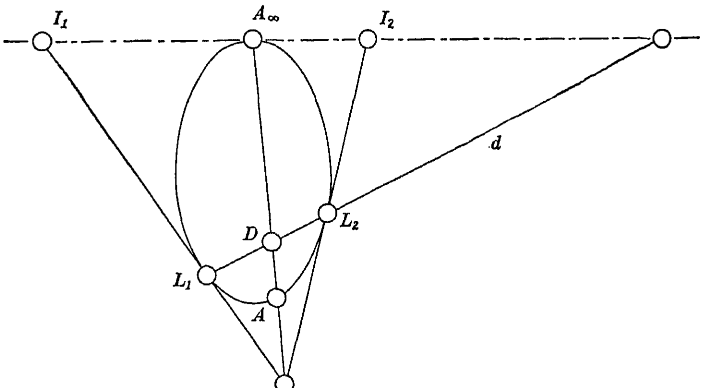


FIG. 66

That the focus, directrix, etc. of a parabola are real may be proved as follows: The transformation from pole to polar with regard to C^2 transforms the absolute involution to an involution of the lines through A_∞ and transforms I_1 and I_2 into $A_\infty L_1$ and $A_\infty L_2$ respectively. The involution in the lines at A_∞ is perspective with an involution among the points of C^2 which has L_1 and L_2 as double points. Hence L_1 and L_2 are conjugate imaginary points. Hence by Theorem 10 the

line d is real. Hence its pole, F, is real. Hence the line a joining F to A_∞ is real, and also the point A.

Since the two tangents to C^2 through F pass through I_1 and I_2, any two conjugate lines through F are perpendicular. Conversely, if the pairs of conjugate lines at any point are orthogonal, the tangents through this point must contain I_1 and I_2 respectively. Hence F is the only such point. Since the tangents through F are imaginary, F is interior to C^2, and hence all real points on d are exterior.

The tangent at A is parallel to d, and hence by the construction of d perpendicular to a. Since the tangent at any other ordinary point of C^2 is not parallel to d, it follows that the line a is the only diameter of C^2 which is perpendicular to its conjugate lines. These and other obvious consequences of the definition may be summarized as follows:

THEOREM 25. *The axis of a parabola is real and is the only diameter perpendicular to all its conjugate lines. The focus of a parabola is real and lies on the axis. The focus is the unique point at which all pairs of conjugate lines are orthogonal. It is interior to the parabola. The directrix is real, is the polar of the focus, and is perpendicular to the axis. All real points of the directrix are exterior to the parabola. The vertex is real and is the mid-point of the focus and the point in which the directrix meets the axis.*

The system of all conics tangent to l_1 and l_2 and to l_∞ at A_∞ forms a range of Type II (§ 47, Vol. I) which consists of all parabolas having F as focus and a as axis. The pairs of tangents to these conics through any real point P of the plane are by the dual of Theorem 20, Chap. V, Vol. I, the pairs of an involution in which PI_1 is paired with PI_2 and PF with PA_∞. The tangents to the two conics of the range which pass through P are the double lines of this involution and hence separate PI_1 and PI_2 harmonically. Thus we have

THEOREM 26. *The parabolas with a fixed focus and axis form a range of Type II. The two parabolas of the range which pass through a given point have orthogonal tangents at this point.*

The tangent to either parabola through P is therefore normal to the other. Since these two lines separate PF and PA_∞ harmonically, we have

THEOREM 27. *The tangent and the normal to a parabola at any point are the bisectors of the pair of lines through this point of which one passes through the focus and the other is a diameter.*

EXERCISES

1. If P is any point of an ellipse, the normal at P is the interior bisector of $\measuredangle F_1PF_2$. If P is any point of a hyperbola, the tangent at P is the interior bisector of $\measuredangle F_1PF_2$.

2. At any nonfocal point in the plane of a conic there is a unique pair of orthogonal lines which are conjugate with regard to the conic. In case of an ellipse or a hyperbola these lines harmonically separate the real foci. In case of a parabola they meet the axis in a pair of points of which the focus is the mid-point.

3 For any point P of an axis of a conic there is a unique point P' on the same axis such that any line through P is orthogonal to its conjugate line through P'. The pairs of points P and P' are pairs of an involution (called a *focal involution*) whose double points are the foci of the conic, or, in case of a parabola, the focus and the point at infinity of the axis. If P and P' are on the minor axis, $\measuredangle PF_1P'$ is a right angle. If the conic is a parabola, F is the mid-point of the pair PP'.

4. Of two confocal central conics having a real point in common, one is an ellipse and the other a hyperbola.

5. The tangents at the points in which a conic is met by a line through a focus meet on the corresponding directrix.

6. If two conics have a focus in common, the poles with regard to the two conics of any line through this focus are collinear with the focus.

7. Let P be any point of a conic, and Q the point in which the tangent at P meets a directrix. If F is the corresponding focus, $\measuredangle PFQ$ is a right angle.

8. If a circle passes through the two real foci and a point P of a conic, it will have the two points in which the tangent and normal at P cut the other axis as extremities of a diameter.

9. If a variable tangent meets two fixed tangents in points P and Q respectively, and F is a focus, the measure of $\measuredangle PFQ$ is constant.

10. Let t_1 and t_2 be two tangents of a central conic meeting in a point T; the pair of lines t_1, TF_1 is congruent to the pair TF_2, t_2.

11. The line joining the focus to the point of intersection of two tangents to a parabola makes with either tangent the same angle that the other tangent makes with the axis.

12. Let p be a variable tangent of a parabola, and P a point of p such that the line PF makes a constant angle with p. The locus of P is a tangent to the parabola.

13. The foci of all parabolas inscribed in a triangle lie on a circle.

14. A circle circumscribed to a triangle which is circumscribed to a parabola passes through the focus.

15. The circles circumscribing four triangles whose sides form a complete

16. Let P be any point coplanar with, but not on an axis of, a conic C^2. The lines which are at once perpendicular to and conjugate with regard to C^2 to the lines through P are the tangents of a parabola (the *Steiner parabola*). The axes of C^2 are tangents of this parabola.

17. If P and P' are a pair of one focal involution of a central conic, and Q and Q' a pair of the other, P, P', Q, Q' are on an equilateral hyperbola, which may degenerate into a pair of orthogonal lines.

18. Given five points of a conic, construct by ruler and compass the center, the axes, the vertices, the foci, and the directrices. Construct the same elements when five tangents are given.

82. Eccentricity of a conic. Let F be a real focus, and d the corresponding directrix, of a conic C^2 which is not a circle. Let a be the major axis of C^2, and h the line parallel to d such that if a meets d in a point D, and h in a point H, D is the mid-point of the pair FH. Then d is the vanishing line (§ 43) of the harmonic homology Γ with F as center and h as axis.

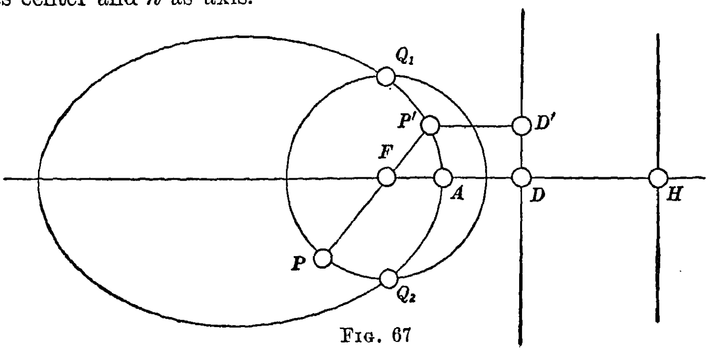


FIG. 67

Since F is a focus, the tangents to C^2 through F pass also through the circular points. Hence the transformation Γ changes C^2 into a circle K^2 with F as center. Now if P is any point of the circle, P' the point of C^2 to which P is transformed by Γ, and D' the point in which the line through P' parallel to FD meets d, it follows by Cor. 2, Theorem 21, Chap. III, that

$$\frac{\text{Dist }(P'F)}{\text{Dist }(P'D')} = \frac{\text{Dist }(PF)}{\text{Dist }(FD)}.$$

Since Dist (PF) and Dist (FD) are constants, it follows that

THEOREM 28. DEFINITION. *The ratio of the distances of a point of a conic to a focus and to the corresponding directrix is a constant*

The conic C^2 is a parabola if and only if the circle K^2 is tangent to d, the vanishing line of Γ. In this case

$$\text{Dist}\,(FD) = \text{Dist}\,(PF),$$

and hence the eccentricity is unity. The conic C^2 is a hyperbola if and only if K^2 meets d in real points. In this case

$$\text{Dist}\,(FD) < \text{Dist}\,(PF),$$

and hence the eccentricity is greater than one. Applying a like remark to the ellipse we have

THEOREM 29. *A conic section is an ellipse, hyperbola, or parabola according as its eccentricity is less than, greater than, or equal to unity.*

A circle is said to have eccentricity zero, because if P and F be held constant, and D be moved so as to increase FD without limit, the ratio $\text{Dist}\,(PF)/\text{Dist}\,(FD)$ approaches zero.

The eccentricity of a hyperbola or an ellipse is evidently the same relatively to either of its real foci, because the two foci and the corresponding directrices are interchangeable by an orthogonal line reflection whose axis is the minor axis of the conic.

As an immediate corollary of the definition of eccentricity we have

THEOREM 30. *Two real conics are similar if and only if they have the same eccentricity.*

On comparing this theorem with Theorem 20, it is evident that the eccentricity is a function of the cross ratio of the double points of the absolute involution and the points in which the conic meets l_∞. As an example of this relation we have (by comparison with § 72) the theorem that any two hyperbolas whose asymptotes make equal angles have the same eccentricity. The formula connecting the eccentricity of a hyperbola with the angular measure of its asymptotes is given in Ex. 7, below, and the formula for the eccentricity in terms of the cross ratio referred to in Theorem 20 is given in Ex. 9.

Since a real focus of any conic is an interior point, the line through a real focus (e.g. F_2, fig. 64) perpendicular to the principal axis meets the conic in two points, Q_1Q_2. The number $\text{Dist}\,(Q_1Q_2)$ is evidently the same for both foci of an ellipse or hyperbola, and hence is a fixed

DEFINITION. The number $p = \text{Dist}(Q_1Q_2)$ is called the *parameter*, or *latus rectum*, of the conic C^2.

In the following exercises e will denote the eccentricity and p the parameter of any conic. For an ellipse or hyperbola a denotes $\text{Dist}(OA_1)$ and c denotes $\text{Dist}(OF_1)$. For an ellipse b denotes $\text{Dist}(OB_1)$. For a hyperbola b denotes $\sqrt{c^2 - a^2}$.

In all cases a radical sign indicates a *positive* square root.

EXERCISES

1. If P is any point of an ellipse, $\text{Dist}(F_1P) + \text{Dist}(F_2P) = 2a$.

2. If P is any point of a hyperbola, $\text{Dist}(F_1P) - \text{Dist}(F_2P) = \pm 2a$.

3. In an ellipse $\text{Dist}(B_1F_1) = a$ and $a^2 = b^2 + c^2$.

4. $\text{Dist}(A_1F_1) \cdot \text{Dist}(F_1A_2) = b^2$.

5. In an ellipse or hyperbola $e = \dfrac{c}{a}$ and $p = \dfrac{2b^2}{a}$.

6. In a parabola $\text{Dist}(AF) = p/4$.

7. The measure θ (§ 67) of the pair of asymptotes of a hyperbola is determined by the equation

$$\cos\theta = 1 - \frac{2}{e^2}.$$

8. For an equilateral hyperbola $e = \sqrt{2}$.

9. The cross ratio $\mathrm{R}(C_1C_2, I_1I_2) = k^2$ referred to in Theorem 20 is connected with the eccentricity by the relation

$$e^2 = \frac{4k}{1 + 2k + k^2}$$

in case of an ellipse, and by $e^2 = \dfrac{-4k}{1 - 2k + k^2}$

in case of a hyperbola.

10. Let A^2 and B^2 be the circles with O as center and passing through the vertices A_1 and B_1, respectively, of an ellipse, and let a variable ray making an angle of measure θ with the ray OA meet these circles in X and Y respectively. Then the line through Y parallel to OA_1 meets the line through X parallel to OB_1 in a point P of the conic. If x and y are the coördinates of P relative to the axes of the conic,

$$x = a\cos\theta, \quad y = b\sin\theta.$$

θ is called the *eccentric anomaly* of the point P.

11. Relative to a nonhomogeneous coördinate system in which the principal axis of a conic is the x-axis, and the tangent at a vertex the y-axis, the equation of a parabola, ellipse, and hyperbola, respectively, can be put in the form

$$y^2 = px,$$

$$y^2 = px - \frac{p}{2a}x^2,$$

$$y^2 = px + \frac{p}{2a}x^2.$$

12. Relative to the asymptotes as axes, the equation of a hyperbola may be written

$$xy = \frac{a^2 + b^2}{4}.$$

13. Relative to any pair of conjugate diameters as axes, an ellipse has the equation

$$\frac{x^2}{a'^2} + \frac{y^2}{b'^2} = 1,$$

and a hyperbola,

$$\frac{x^2}{a'^2} - \frac{y^2}{b'^2} = 1.$$

If A' is a point in which the x-axis meets the conic, Dist $(OA') = a'$. In the case of an ellipse, if B' is one of the points in which the y-axis meets the conic, Dist $(OB') = b'$.

14. The measure of the ordered point triads $OA'B'$ is a constant.

15. The numbers a' and b' satisfy the conditions $a'^2 + b'^2 = a^2 + b^2$ in case of an ellipse and $a'^2 - b'^2 = a^2 - b^2$ in case of a hyperbola.

16. The equation of a system of confocal central conics relative to a system of nonhomogeneous point coördinates in which the axes of the conics are $x = 0$ and $y = 0$ is

$$\frac{x^2}{a^2 - \lambda} + \frac{y^2}{b^2 - \lambda} = 1,$$

where λ is a parameter. In the homogeneous line coördinates such that $u_1x + u_2y + u_0 = 0$ gives the condition that the point (x, y) be on the line $[u_0, u_1, u_2]$, the equation of a system of confocals is $u_0^2 = (a^2 - \lambda)u_1^2 + (b^2 - \lambda)u_2^2$.

17. Relative to point coördinates in which the origin is the focus, $y = 0$ the axis of the parabolas, and $x = 0$ perpendicular to the axis, the equation of a system of confocal parabolas is

$$y^2 - 2(p - \lambda)x + \lambda(p - \lambda) = 0.$$

In the corresponding homogeneous line coördinates this is (cf. Ex. 16)

$$pu_2^2 - 2u_1u_0 - \lambda(u_1^2 + u_2^2) = 0.$$

83. Synoptic remarks on conic sections. An inspection of the literature will convince one that it would not be practical to include a complete list of the known metric theorems on conic sections in a book like this one. The theorems which we have derived, however, are sufficient to indicate how the rest may be obtained either directly as special cases of projective theorems or as consequences of the focal and affine theorems given in this chapter and Chap. III.

The theorems on conic sections have been classified according to the geometries to which they belong. The most general and elementary which we have considered are those which belong to the proper projective geometry (§ 17), the geometry corresponding to the projec-

of this class are given in Vol. I, particularly in Chaps. V, VIII, X. A second large class contains those theorems which belong to the affine geometry in any proper projective space. These are treated somewhat fully in Chap. III.

The theorems of the class considered in §§ 74, 75 of this chapter belong to the projective geometry of an ordered space. The theorems of § 76 belong to the projective geometry of a real space. Finally, in §§ 80–82 we have been considering theorems of the Euclidean geometry of a real space.

It is quite feasible to make a much finer classification of theorems on conics. This would mean, for example, distinguishing those properties of foci which hold in a parabolic metric geometry in a general space, then those which hold in an ordered space, and then those which are peculiar to the real space.

The theorems which have been under discussion in the remarks above refer in general to figures composed of one conic section and a finite number of points and lines. Theorems regarding more than one conic at a time have not been considered in any considerable number, and the theory of families of conics has not been carried beyond pencils and ranges. For an outline of this subject the reader is referred to the Encyclopädie der Math. Wiss., III C1, §§ 56–90.

EXERCISES

1. The diagonals of the rectangle formed by the tangents at the vertices of an ellipse are conjugate diameters for which $a' = b'$. The angle between this pair of conjugate diameters is less than that between any other pair of conjugate diameters. For this pair of conjugate diameters $a' + b'$ is a maximum. It is a minimum for $a' = a$, $b' = b$.

2. If two orthogonal diameters of a conic meet it in P and Q,

$$\frac{1}{OP^2} + \frac{1}{OQ^2} = \frac{1}{a^2} + \frac{1}{b^2}$$

for an ellipse, and

$$\frac{1}{OP^2} - \frac{1}{OQ^2} = \pm\left(\frac{1}{a^2} - \frac{1}{b^2}\right)$$

for a hyperbola.

3. The locus of a point from which the two tangents to a conic C^2 are orthogonal is a real circle in case C^2 is an ellipse or a hyperbola for which $a > b$; is a pair of conjugate imaginary lines through the center and the circular points in case C^2 is a hyperbola for which $a = b$; is an imaginary circle in case C^2 is a hyperbola for which $a < b$; is the directrix in case C^2 is a parab-

4. A variable tangent to a central conic is met by the lines through a focus ich make a fixed angle with it in the points of a circle. In particular, the us of the foot of a perpendicular from a focus to a tangent is a circle.

5. If t is a variable tangent of a central conic, Dist $(F_1t)\cdot$ Dist $(F_2t)=b^2$. t' is the other tangent parallel to t, Dist $(F_1t)\cdot$ Dist $(F_1t')=b^2$.

6. If F is a focus of a conic and P_1, P_2 the points of intersection of an itrary line through F with the conic,

$$\frac{1}{P_1F}+\frac{1}{FP_2}$$

constant.

7. If the tangent to a conic at a variable point P meets the axes in two nts T_1 and T_2, and the normal at P meets them in N_1 and N_2, then

$$\begin{aligned}\text{Dist }(PT_1)\cdot\text{Dist }(PT_2)&=\text{Dist }(PN_1)\cdot\text{Dist }(PN_2)\\&=\text{Dist }(PF_1)\cdot\text{Dist }(PF_2).\end{aligned}$$

8. There is a unique circle which osculates* a given conic at a given nt P. This is called the *circle of curvature* at P. Its center is called the *ter of curvature* for P and lies on the normal at P.

9. Construct by ruler and compass the center of the circle of curvature in arbitrary point of a given conic.

10. The circle of curvature of a conic C^2 at a point P meets C^2 in one and y one other point, Q. The line PQ is the axis and the point P the center of an ion which transforms K^2 into C^2. The center of curvature is transformed this elation into the center of the involution on C^2 in which the pairs of logonal lines at P meet C^2.

11. The tangent and normal at any point P of a conic C^2 are both tangent he Steiner parabola (Ex. 16, § 81) determined by this point. The point ontact of the normal with the parabola is the center of the circle of curure of C^2 at P, and the point of contact of the tangent with the parabola he pole of the normal with respect to C^2. (For further properties of the le of curvature, cf. Encyclopädie der Math. Wiss., III C1, § 36.)

12. The polar reciprocal of a circle with respect to a circle having a nt O as center is a conic having O as a focus. (A set of theorems related his one will be found in Chap. VIII of the book by J. W. Russell referred n § 73.)

84. **Focal properties of collineations.** The focal properties of conic ions are closely related to a set of theorems on collineations ne of which are given in the exercises below. A good treatnt of the subject is to be found in the Collected Papers of J. S. Smith, Vol. I, p. 545, and further references in the cyclopädie der Math. Wiss., III AB 5, § 9.

Let Π be any real projective collineation which does not leave l_∞ invariant, and let p and q be its vanishing lines; so that $\Pi(p) = l_\infty$ and $\Pi(l_\infty) = q$. If I_1 and I_2 are the circular points, let $\Pi^{-1}(I_1) = P_1$, $\Pi^{-1}(I_2) = P_2$, $\Pi(I_1) = Q_1$, $\Pi(I_2) = Q_2$. By the theorems of § 78 the lines P_1I_1 and P_2I_2 meet in a real point A_1, and P_1I_2 and P_2I_1 meet in a real point A_2. If $\Pi(A_1) = B_1$ and $\Pi(A_2) = B_2$, it is clear that the complete quadrilateral whose pairs of opposite vertices are I_1I_2, P_1P_2, A_1A_2 is transformed into one whose pairs of opposite vertices are Q_1Q_2, I_1I_2, B_1B_2. The following propositions are now easily verifiable, and are stated as exercises.

EXERCISES

1. A_1 is such that any ordered pair of lines meeting at A_1 is transformed by Π into a congruent pair of lines. A_2 is such that any two lines meeting in A_2 are transformed by Π into a symmetric pair of lines. No other points have either of these properties.

2. Every conic having a focus at A_1 or A_2 goes to a conic with a focus at B_1 or B_2 respectively.

3. The range of conics having A_1 and A_2 as foci is transformed by Π into the range of conics with B_1 and B_2 as foci; and this is the only system of confocals which goes into a system of confocals.

4. The pencil of circles with A_1, A_2 as limiting points is transformed by Π into that having B_1, B_2 as limiting points; and these are the only two pencils of circles homologous under Π. The radical axes of the two pencils are the two vanishing lines.

5. If P is any point and $\Pi(P) = P'$, then the ordered point triad A_1PA_2 is similar (but not directly similar) to the ordered point triad $B_1P'B_2$.

6. At a point of a Euclidean plane there is in general one and only one pair of perpendicular lines which is transformed into a pair of perpendicular lines by a given affine collineation.

7. In any two projective pencils of lines there is a pair of corresponding orthogonal pairs of lines. The line pairs which are homologous with congruent line pairs form an involution.

8. Any projective collineation which does not leave l_∞ invariant is expressible as a product of a displacement and a homology.

85. Homogeneous quadratic equations in three variables. Reversing the process which is common in analytic geometry, it is possible to derive certain classes of algebraic theorems from the theory of conic sections. We shall illustrate this process in a few important

The general homogeneous equation of the second degree can be written in the form

$$
(6) \qquad \begin{aligned} & a_{00}x_0^2 + a_{01}x_0x_1 + a_{02}x_0x_2 \\ & + a_{10}x_1x_0 + a_{11}x_1^2 + a_{12}x_1x_2 \\ & + a_{20}x_2x_0 + a_{21}x_2x_1 + a_{22}x_2^2 = 0, \end{aligned}
$$

where $a_{ij} = a_{ji}$. Let us first suppose that

$$
(7) \qquad A \equiv \begin{vmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{vmatrix} \neq 0.
$$

In § 98, Vol. I, it has been shown, from the point of view of general projective geometry, that every projective polarity is represented by a bilinear equation of the form

$$
(8) \qquad \begin{aligned} & a_{00}x_0x_0' + a_{01}x_0x_1' + a_{02}x_0x_2' \\ & + a_{10}x_1x_0' + a_{11}x_1x_1' + a_{12}x_1x_2' \\ & + a_{20}x_2x_0' + a_{21}x_2x_1' + a_{22}x_2x_2' = 0, \end{aligned}
$$

where $a_{ij} = a_{ji}$ and where $A \neq 0$.

It was also shown that every bilinear equation of this form, subject to the condition $A \neq 0$, represents a polarity; that the equation in point coördinates of the fundamental conic of the polarity is (6), which is obtained from (8) by setting $x_i' = x_i$; and that the equation of this conic in line coördinates is

$$
(9) \qquad A_{ij}u_iu_j = 0,
$$

where A_{ij} is the cofactor of a_{ij} in A.

The coefficients a_{ij} are elements of the geometric number system. Therefore in the case of the real plane they are real numbers, and we have

THEOREM 31. *Every equation of the form* (6) *with real coefficients such that* $a_{ij} = a_{ji}$ *and* $A \neq 0$ *represents a conic whose polar system transforms real points into real lines. Conversely, every conic with regard to which real points have real polars has an equation of the form* (6) *with real coefficients such that* $a_{ij} = a_{ji}$ *and* $A \neq 0$.

In § 79 we have seen that any conic having a real polar system is in one of two classes, and that any two conics of the same class are projectively equivalent. Now it is obvious that

$$x_0^2 + x_1^2 + x_2^2 = 0 \tag{10}$$

is the equation of an imaginary conic, and that

$$x_0^2 + x_1^2 - x_2^2 = 0 \tag{11}$$

is the equation of a real conic. Hence we have

THEOREM 32. *Any quadratic equation in three homogeneous variables whose discriminant A does not vanish is reducible by real linear homogeneous transformation of the variables to the form* (10) *or to the form* (11).

Algebraic criteria to determine whether a given conic C^2 whose equation is in the form (6) belongs to one or the other of these classes may easily be determined by the aid of simple geometric considerations. In case C^2 contains no real points, the line $x_0 = 0$ has no real point in common with it, and the point $u_1 = 0$ (which is on the line $x_0 = 0$) is on no real tangent to it. On the other hand, if the line $x_0 = 0$ contained no real point of C^2, and C^2 were real, this line would consist entirely of exterior points, and hence there would be a tangent to C^2 through the point $u_1 = 0$. Hence a pair of necessary and sufficient conditions that C^2 contain no real points are (1) $x_0 = 0$ is on no point of $C,^2$ and (2) $u_1 = 0$ is on no tangent of C^2.

Substituting $x_0 = 0$ and $x_0' = 0$ in (8), we have the equation of an involution

$$\begin{aligned} a_{11}x_1x_1' &+ a_{12}x_1x_2' \\ + a_{21}x_2x_1 &+ a_{22}x_2x_2' = 0, \end{aligned} \tag{12}$$

which, by § 4, is elliptic if and only if $A_{00} > 0$. By a dual argument applied to (9), the necessary and sufficient condition that there be no real tangents to C^2 through the point $u_1 = 0$ is

$$\begin{vmatrix} A_{00} & A_{02} \\ A_{20} & A_{22} \end{vmatrix} > 0. \tag{13}$$

By a well-known theorem on determinants (or a simple computation) this reduces to

Hence we have

THEOREM 33. *The imaginary conics are those for which*

$$A_{00} > 0 \text{ and } a_{11} \cdot A > 0,$$

and the real ones are those for which not both of these conditions are satisfied and for which $A \neq 0$.

In these conditions it is obvious that A_{00} and a_{11} may be replaced by A_{ii} and a_{jj}, where $i, j = 0, 1, 2$, provided that $i \neq j$.

Let us now investigate the cases where $A = 0$, and first the case in which not all the cofactors A_{00}, A_{11}, A_{22} are zero. To fix the notation, suppose that $A_{00} \neq 0$. Then the bilinear equation (8) is satisfied by $x_0 = A_{00}$, $x_1 = A_{01}$, $x_2 = A_{02}$, no matter what values are taken by x'_0, x'_1, x'_2. Hence in this case (8) determines a transformation, Γ, of all the points (x'_0, x'_1, x'_2) distinct from (A_{00}, A_{01}, A_{02}) into lines through (A_{00}, A_{01}, A_{02}). A collineation which transforms (A_{00}, A_{01}, A_{02}) to $(1, 0, 0)$ must reduce (8) to

$$(14) \qquad \begin{aligned} &b_{11}x_1x'_1 + b_{12}x_1x'_2 \\ &+ b_{21}x_2x'_1 + b_{22}x_2x'_2 = 0 \end{aligned} \qquad b_{12} = b_{21}.$$

It is to be noted that

$$\begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} \neq 0,$$

because if this determinant vanished, Γ would transform all points (x'_0, x'_1, x'_2) into a single line, and hence A_{00} would vanish. Hence Γ transforms any point (x'_0, x'_1, x'_2) into the line paired in a certain involution with the line joining (x'_0, x'_1, x'_2) to (A_{00}, A_{01}, A_{02}). The double lines of the involution must satisfy the quadratic equation (6).

Comparing with the definitions in § 45, Vol. I, we have that when $A = 0$ and not all the cofactors A_{00}, A_{11}, A_{22} are zero, (6) represents a degenerate conic consisting of two distinct lines and that (8) represents the polar system of the conic. Since the lines represented by (6) are the double lines of a real involution, they are either real or a pair of conjugate imaginaries. In the first case (6) can evidently be transformed by a collineation to

$$(15) \qquad x_1^2 - x_2^2 = 0,$$

and in the second case to

(just as in the nondegenerate case) if and only if $A_{ii} > 0$, and real points if and only if $A_{ii} \leqq 0$. Hence the case where (6) represents a pair of real lines occurs if and only if $A_{ii} \leqq 0$, for $i = 0, 1, 2$.

Finally, suppose that $A_{00} = A_{11} = A_{22} = A = 0$. In view of the identity,

$$A_{ii}A_{jj} - A_{ij}^2 \equiv a_{kk} \cdot A, \qquad (i \neq j \neq k \neq i) \tag{17}$$

this implies that all the cofactors A_{ij} are zero, and hence that (8) represents the same line, no matter what values are substituted for x_0', x_1', x_2'. Hence (6) represents a single real line (i.e. two coincident real lines), and the polar system (8) transforms all points not on this line into this line. If this line be transformed to $x_1 = 0$, (6) obviously becomes

$$x_1^2 = 0. \tag{18}$$

A degenerate point conic is two distinct or coincident lines. These may always be represented by a quadratic equation which is a product of two linear ones. For such a quadratic $A = 0$, because if $A \neq 0$, the equation has been seen to represent a nondegenerate conic. Hence the theory of degenerate point conics is equivalent to that of homogeneous quadratic equations for which $A = 0$.

The complete projective classification of conics, degenerate or not, may now be stated as an algebraic theorem in the form:

THEOREM 34. *Any homogeneous quadratic equation in three variables may be reduced by a real linear homogeneous transformation,*

$$x_i' = \sum_{j=0}^{2} \alpha_{ij} x_j, \qquad (i = 0, 1, 2), |\alpha_{ij}| \neq 0 \tag{19}$$

to one of the normal forms (10), (11), (16), (15), (18). *The criteria which determine to which one of these forms an equation* (6) *is reducible may be summarized in the following table:*

$A \neq 0$		$A = 0$		
IMAGINARY CONIC	REAL CONIC	IMAGINARY LINE PAIR	REAL LINE PAIR	COINCIDENT REAL LINE PAIR
$a_{11}A > 0$	$a_{11}A \leqq 0$	$A_{00} > 0$	$A_{00} < 0$	$A_{00} = 0$
		or $A_{11} > 0$	or $A_{11} < 0$	$A_{11} = 0$
$A_{00} > 0$	or $A_{00} \leqq 0$	or $A_{22} > 0$	or $A_{22} < 0$	$A_{22} = 0$

Since the algebraic expressions in the above criteria determine conditions on the conic which are independent of the choice of coördinates and thus are invariant under the projective group, it is natural to inquire whether they are algebraic invariants in the sense of § 90, Vol. I. A direct substitution will readily verify that A is a relative invariant of (6).

Suppose we regard the coefficients of (6) as homogeneous coördinates $(a_{00}, a_{11}, a_{22}, a_{01}, a_{10}, a_{12})$ of a point in a five-dimensional space. Then $A = 0$ determines a certain cubic locus in this space the points on which represent degenerate conics. Now if there were any other invariant of (6) under the projective group, say $\phi(a_{ij})$, the equation $\phi(a_{ij}) = 0$ would represent a locus in this five-dimensional space. But since each nondegenerate conic is projectively equivalent to every other nondegenerate conic, this locus would have to be contained in the locus of $A = 0$. From this it can be proved, by the general theory of loci represented by algebraic equations, that the locus of $\phi(a_{ij}) = 0$ coincides with that of $A = 0$, and that hence $\phi(a_{ij})$ is rationally expressible in terms of A. Thus A is essentially the only invariant of (6) under the projective group.

The question, however, arises whether there are not other rational functions of the coefficients of (6) which are invariant whenever $A = 0$. If there were such a function, say $\phi(a_{ij})$, the conics for which $\phi(a_{ij}) = 0$ would be a subclass of the degenerate conics which is transformed into itself by all complex projective collineations. The only class of this sort consists of the coincident line pairs which are given by *two* conditions, $A_{00} = 0$, $A_{11} = 0$. In view of the theorem that a locus represented by two independent algebraic equations cannot be the complete locus of a single algebraic equation, this shows that there is no other invariant of (6) even for the cases in which $A = 0$.

This reasoning could be expressed still more briefly by saying that, while the set of all conics is a five-parameter family, and the set of degenerate conics a four-parameter family given by one condition, the only invariant subset of the degenerate conics is the two-parameter set of coincident line pairs which have to be given by two conditions and so cannot correspond to a single invariant in addition to A.

EXERCISES

1. In case $A = 0$, the lines represented by (6) intersect in the point $(\sqrt{A_{00}}, \sqrt{A_{11}}, \sqrt{A_{22}})$, unless the three cofactors A_{ii} vanish, in which case (6) represents the coincident line pair

$$(\sqrt{a_{00}}x_0 + \sqrt{a_{11}}x_1 + \sqrt{a_{22}}x_2)^2 = 0.$$

2. In case (6) represents a pair of distinct lines, (9) represents their point of intersection counted twice. In case (6) represents a pair of coincident

86. Nonhomogeneous quadratic equations in two variables. The affine theory of point conics corresponds to the theory of

$$(20)\qquad \begin{aligned} &a_{00} + a_{01}x + a_{02}y \\ &+ a_{10}x + a_{11}x^2 + a_{12}xy \\ &+ a_{20}y + a_{21}yx + a_{22}y^2 = 0, \end{aligned}$$

where the a_{ij}'s satisfy the same conditions as in the last section. The theorem that any nondegenerate conic is an imaginary ellipse, real ellipse, hyperbola, or parabola, and that any two conics of the same class are equivalent under the affine group, translates into the following: Any quadratic equation in two variables, for which $A \neq 0$, is transformable by a transformation of the form

$$(21)\qquad \begin{aligned} x' &= a_1x + b_1y + c_1, \\ y' &= a_2x + b_2y + c_2, \end{aligned} \qquad \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0,$$

into one of the following four forms:

$$(22)\qquad x^2 + y^2 + 1 = 0,$$

$$(23)\qquad x^2 + y^2 - 1 = 0,$$

$$(24)\qquad x^2 - y^2 - 1 = 0,$$

$$(25)\qquad x^2 + y = 0.$$

To know this it is merely necessary to observe that these equations represent conics of the four types respectively.

The criteria to determine in which class a given conic C^2 belongs may be inferred from the discussion in the last section if we set $x = x_1/x_0$ and $y = x_2/x_0$. It is then evident that $A_{00} > 0$ for an ellipse, $A_{00} = 0$ for a parabola, and $A_{00} < 0$ for a hyperbola. Hence the affine classification of cases where $A \neq 0$ may be summarized in the following table:

$A \neq 0$

IMAGINARY ELLIPSE	REAL ELLIPSE	HYPERBOLA	PARABOLA
$A_{00} > 0$ $a_{11}A > 0$	$A_{00} > 0$ $a_{11}A \leqq 0$	$A_{00} < 0$	$A_{00} = 0$

The cases where $A = 0$ correspond, as in the last section, to degenerate conics. Geometrically the types of figures are obvious, and to obtain the algebraic criteria we need only combine with considera-

$A = 0$

Conjugate Imaginary Lines		Distinct Real Lines			Coincident Real Lines	
Concurrent at ordinary point	Parallel pair	Concurrent at ordinary point	Parallel pair	One at infinity	Ordinary	At infinity
$A_{00} > 0$	$A_{00} = 0$, $A_{11} > 0$ or $A_{22} > 0$	$A_{00} < 0$	$A_{00} = 0$, $A_{11} < 0$ or $A_{22} < 0$;		$A_{00} = A_{11} = A_{22} = 0$;	
			$a_{11} \neq 0$ or $a_{22} \neq 0$	$a_{11}=a_{22}=0$	$a_{11} \neq 0$ or $a_{22} \neq 0$	$a_{11}=a_{22}=0$

As normal forms for the first six cases we may take

(26) $$x^2 + y^2 = 0,$$

(27) $$x^2 + 1 = 0,$$

(28) $$x^2 - y^2 = 0,$$

(29) $$x^2 - 1 = 0,$$

(30) $$x = 0,$$

(31) $$x^2 = 0.$$

The case of coincident real lines at infinity does not correspond to any equation in nonhomogeneous coördinates.

Summarizing these results we have the following algebraic theorem:

THEOREM 35. *Any quadratic equation in two variables may be reduced to one and only one of the normal forms* (22)–(31) *by a transformation of the form* (21). *The normal form to which it is reducible is determined by the criteria in the two tables above.*

The question of invariants of (20) under the affine group may be investigated in the manner indicated for the corresponding projective problem in the fine print at the end of the last section. The results of such an investigation are given in the exercises below.

There are no absolute invariants of conics under the projective

1. A and A_{00} are invariants of (20) under the affine group.

2. In case $A = A_{00} = 0$, A_{11}/a_{22} and A_{22}/a_{11} are invariants of (20) under the affine group.

3. The homogeneous coördinates of the center of (20) are (A_{00}, A_{01}, A_{02}).

4. If $A_{00} \neq 0$, the translation $\bar{x} = x - \dfrac{A_{10}}{A_{00}}$, $\bar{y} = y - \dfrac{A_{20}}{A_{00}}$ transforms (20) into

$$a_{11}\bar{x}^2 + 2\,a_{12}\bar{x}\bar{y} + a_{22}\bar{y}^2 + \frac{A}{A_{00}} = 0.$$

5. If $A \neq 0$ and $A_{00} \neq 0$, the asymptotes of (20) are given by the equation

$$a_{11}\bar{x}^2 + 2\,a_{12}\bar{x}\bar{y} + a_{22}\bar{y}^2 = 0.$$

6. Any diameter of a parabola is parallel to $a_{11}x + a_{12}y = 0$ and tc $a_{12}x + a_{22}y = 0$.

87. Euclidean classification of point conics. With respect to a non-homogeneous coördinate system in which the pair of lines $x = 0$ and $y = 0$ is orthogonal and bisected by the lines $x = y$ and $x = -y$, the transformations of the Euclidean group take the form (21) subject to the conditions

$$a_1^2 + a_2^2 = b_1^2 + b_2^2, \qquad a_1b_1 + a_2b_2 = 0, \tag{32}$$

and the displacements are subject to the additional condition

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 1. \tag{33}$$

Since any ellipse or hyperbola is congruent to one whose principal axes are $x = 0$ and $y = 0$, and since any parabola is congruent to a parabola with the origin as vertex and $y = 0$ as its principal axis, it follows that any conic is congruent to a conic having one of the following equations:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + 1 = 0, \tag{34}$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0, \tag{35}$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0, \tag{36}$$

$$y^2 - px = 0. \tag{37}$$

The normal forms to which degenerate point conics can be reduced by displacements are evident when one recalls that two pairs of non-

the circular points and that two pairs of parallel lines are congruent if the lines of each pair are the same distance apart.* By comparison with the second table ($A = 0$) in § 86 we find

$$(38) \qquad \frac{x^2}{a^2} + y^2 = 0,$$

$$(39) \qquad x^2 + c^2 = 0,$$

$$(40) \qquad \frac{x^2}{a^2} - y^2 = 0,$$

$$(41) \qquad x^2 - c^2 = 0,$$

$$(42) \qquad x = 0,$$

$$(43) \qquad x^2 = 0.$$

The group of displacements is extended to the group of similarity transformations by adjoining transformations of the form

$$(44) \qquad \begin{aligned} x' &= kx, \\ y' &= ky, \end{aligned} \qquad k \neq 0.$$

Transformations of this sort will reduce the equations (34) – (43) to normal forms in which b, c, and p are all unity.

The criteria for determining to which of these normal forms a conic is reducible under the group of displacements or that of similarity transformations are the same as those already found for the affine group. Two conics whose equations can be reduced to the same normal form are evidently equivalent under the group of displacements if and only if they determine the same values for a and b or c or p, and under the Euclidean group if they determine the same value for a. The numbers a, b, c, p are evidently absolute invariants of the corresponding conics under the group of displacements, and a in (38) and (40) also under the Euclidean group.

The problem of determining a, b, c, p in terms of the coefficients of (20) presents no special difficulty, and will be left to the reader to be considered in connection with the exercises below and those at the end of the next section.

When b, c, p are all unity, a is a function of the eccentricity given by the equations in Exs. 7 and 9, § 82. The same reference gives the connection between the eccentricity and the invariant $\sqrt{-A_{00}}/(a_{11} + a_{22})$.

* The distance apart is the distance of an arbitrary point on one of the parallel lines from the other line. The formula for distance is applied to the case of a pair

1. If $A \neq 0$ and $A_{00} \neq 0$, the angular measure of the asymptotes is θ, where

$$\tan\theta = \frac{2\sqrt{-A_{00}}}{a_{11}+a_{22}}.$$

Moreover, $$\theta = -\frac{i}{2}\log \mathrm{R}\,(C_1C_2,\ I_1I_2),$$

where C_1 and C_2 are the points in which the conic meets l_∞, and I_1 and I_2 are the circular points. If $A=0$ and $A_{00}\neq 0$, these formulas give the angular measure of the lines represented by (20). Derive from this the formula for a in (38) and (40) in terms of the coefficients of (20).

2. A_{00} and $a_{11}+a_{22}$ are absolute invariants of (20) under the group of displacements, and $\sqrt{-A_{00}}/(a_{11}+a_{22})$ under the Euclidean group. If $A \neq 0$ and $a_{11}+a_{22}=0$, (20) represents an equilateral hyperbola; if $A=0$ and $a_{11}+a_{22}=0$, it represents a pair of orthogonal lines or l_∞ and an ordinary line.

3. If $A \neq 0$ and $A_{00}\neq 0$, the axes of (20) are

$$a_{12}(\bar{x}^2+\bar{y}^2)+(a_{22}-a_{11})\,\bar{x}\bar{y}=0,$$

where $\bar{x}$ and $\bar{y}$ are defined as in Ex. 4, § 86.

4. For an ellipse the constants a and b are $\sqrt{\dfrac{-A}{A_{00}\lambda_1}}$ and $\sqrt{\dfrac{-A}{A_{00}\lambda_2}}$, where λ_1 and λ_2 are the roots of

(45) $$\lambda^2-(a_{11}+a_{22})\lambda+A_{00}=0;$$

and for a hyperbola a and ib are $\sqrt{\dfrac{-A}{A_{00}\lambda_1}}$ and $\sqrt{\dfrac{-A}{A_{00}\lambda_2}}$. The discriminant of (45) is $(a_{11}-a_{22})^2+4a_{12}^2$.

5. If $A\neq 0$ and $A_{00}=0$, the parabola (20) touches l_∞ at $(0,\ a_{12},\ -a_{11})$, which is the same as $(0,\ a_{22},\ -a_{12})$. The axis is

(46) $$a_{11}x+a_{12}y+\frac{a_{01}a_{11}+a_{02}a_{12}}{a_{11}+a_{22}}=0.$$

88. Classification of line conics. The projective classification of line conics is entirely dual to that of point conics and so need not be considered separately. The affine classification, however, corresponds to a new algebraic problem. If the line coördinates are chosen so that

$$u_0x_0+u_1x_1+u_2x_2=0$$

is the condition that the point (x_0, x_1, x_2) be on the line $[u_0, u_1, u_2]$, the point coördinates being the same as already used, we have the problem of reducing equations of the form (9) to normal forms by means of transformations of the form

(47) $$\begin{aligned} u_0' &= d_0u_0+d_1u_1+d_2u_2,\\ u_1' &= \qquad\ \ +b_2u_1-a_2u_2,\\ u_2' &= \qquad\ \ -b_1u_1+a_1u_2, \end{aligned}\qquad \begin{vmatrix} a_1 & b_1\\ a_2 & b_2\end{vmatrix}\neq 0.$$

These are the transformations which leave the line [1, 0, 0] invariant. If

$$d_0 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}, \quad d_1 = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}, \quad \text{and} \quad d_2 = \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix},$$

(47) is the same collineation as (21).

The affine classification of nondegenerate line conics is of course the same as that of nondegenerate point conics. To express the criteria in terms of the equation (9) regarded as given primarily,* let us write

$$(48) \qquad \alpha \equiv \begin{vmatrix} A_{00} & A_{01} & A_{02} \\ A_{10} & A_{11} & A_{12} \\ A_{20} & A_{21} & A_{22} \end{vmatrix},$$

where the A_{ij}'s are the coefficients of (9), and let α_{ij} denote the cofactor of A_{ij} in α. The point conic associated with (9) must have the equation

$$(49) \qquad \sum \alpha_{ij} x_i x_j = 0.$$

By the criteria already worked out, this is an ellipse, hyperbola, or parabola according as the value of

$$\begin{vmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{vmatrix} \equiv A_{00} \cdot \alpha$$

is greater than, less than, or equal to zero; and, in the case of an ellipse, real or imaginary according as $\alpha_{11} > 0$ or $\alpha_{11} \leqq 0$. Thus we have

$$\alpha \neq 0$$

IMAGINARY ELLIPSE	REAL ELLIPSE	HYPERBOLA	PARABOLA
$\alpha \cdot A_{00} > 0$ $\alpha_{11} > 0$	$\alpha \cdot A_{00} > 0$ $\alpha_{11} \leqq 0$	$\alpha \cdot A_{00} < 0$	$A_{00} = 0$

The normal forms for these four classes are respectively

$$(50) \qquad u_0^2 + u_1^2 + u_2^2 = 0,$$

$$(51) \qquad u_0^2 - u_1^2 - u_2^2 = 0,$$

$$(52) \qquad u_0^2 - u_1^2 + u_2^2 = 0,$$

$$(53) \qquad u_1^2 - u_2 = 0.$$

The projective classification of degenerate line conics is dual to that of degenerate point conics, and therefore yields the following three cases: (1) two distinct real points, $\alpha = 0$, $\alpha_{ii} \leqq 0$, one at least

* Instead of in terms of the coefficients of (6).

of α_{00}, α_{11}, α_{22} being different from zero; (2) coincident real points, $\alpha=\alpha_{11}=\alpha_{22}=\alpha_{33}=0$; (3) conjugate imaginary points, $\alpha=0$, $\alpha_{ii}>0$ for at least one value of i.

For the affine classification let us observe that since [1, 0, 0] is the line at infinity, the condition that at least one factor of (9) represent a point at infinity is $A_{00}=0$. The following criteria are now evident.

$$\alpha = 0$$

<table>
<tr><th colspan="2">Conjugate Imaginary Points</th><th colspan="3">Distinct Real Points</th><th colspan="2">Coincident Real Points</th></tr>
<tr><th>Ordinary</th><th>At infinity</th><th>Both ordinary</th><th>One ordinary</th><th>Both at infinity</th><th>Ordinary</th><th>At infinity</th></tr>
<tr><td>$\alpha_{11}>0$ or $\alpha_{22}>0$</td><td>$\alpha_{00}>0$
$\alpha_{11}=\alpha_{22}=0$</td><td colspan="2">$\alpha_{11}<0$ or $\alpha_{22}<0$</td><td>$\alpha_{00}<0$
$\alpha_{11}=0$
$\alpha_{22}=0$</td><td colspan="2">$\alpha_{00}=\alpha_{11}=\alpha_{22}=0$</td></tr>
<tr><td></td><td></td><td>$A_{00}\neq 0$</td><td>$A_{00}=0$</td><td></td><td>$A_{00}\neq 0$</td><td>$A_{00}=0$</td></tr>
</table>

The normal forms for these cases are respectively

(54) $$u_0^2+u_1^2=0,$$

(55) $$u_1^2+u_2^2=0,$$

(56) $$u_0^2-u_1^2=0,$$

(57) $$u_0u_1=0,$$

(58) $$u_1u_2=0,$$

(59) $$u_0^2=0,$$

(60) $$u_1^2=0.$$

EXERCISES

1. The two pairs of foci of (9) are the degenerate conics of the range

(61) $$\begin{aligned} &A_{00}u_0^2+A_{01}u_0u_1+A_{02}u_0u_2 \\ &+A_{10}u_1u_0+(A_{11}-\rho)\,u_1^2+A_{12}u_1u_2 \\ &+A_{20}u_2u_0+A_{21}u_2v_1+(A_{22}-\rho)\,u_2^2=0, \end{aligned}$$

which are given by the values of ρ satisfying

(62) $$A_{00}\rho^2-(\alpha_{11}+\alpha_{22})\,\rho+\alpha=0.$$

The discriminant of this quadratic is $(\alpha_{11}-\alpha_{22})^2+4\,\alpha_{12}^2$.

2. In case $\alpha = 0$ and $A_{00} \neq 0$, the distance between the points represented by (9) is

$$\frac{2\sqrt{-(\alpha_{11}+\alpha_{22})}}{A_{00}}.$$

3. The normal forms for line conics under the group of displacements are

$$u_0^2 + a^2u_1^2 + b^2u_2^2 = 0, \tag{63}$$
$$u_0^2 - a^2u_1^2 - b^2u_2^2 = 0, \tag{64}$$
$$u_0^2 - a^2u_1^2 + b^2u_2^2 = 0, \tag{65}$$
$$4\,u_0u_1 + pu_2^2 = 0, \tag{66}$$
$$u_0^2 + k^2u_1^2 = 0, \tag{67}$$
$$u_1^2 + c^2u_2^2 = 0, \tag{68}$$
$$u_0^2 - k^2u_1^2 = 0, \tag{69}$$
$$u_0u_1 = 0, \tag{70}$$
$$u_1^2 - c^2u_2^2 = 0, \tag{71}$$
$$u_0^2 = 0, \tag{72}$$
$$u_1^2 = 0. \tag{73}$$

Here a, b, p have the same significance as in (34)–(37); $2\,ki$ is the distance between the two points represented by (67); $2\,k$ is the distance between the two points represented by (69); c is expressible in terms of the cross ratio of the circular points and the two points represented by (68) or (71).

*** 89. Polar systems.** The theorems on the classification of conics (§ 79) may be regarded as completing the discussion of projective polar systems in a real plane. There is, however, a certain amount of interest in making the discussion of polar systems without the intervention of complex elements, and basing it entirely on the most elementary theorems about order relations. This treatment will hold good for a projective space satisfying Assumptions A, E, S, P.

THEOREM 36. *In any projective polar system in an ordered plane the involutions of conjugate points on the sides of a self-polar triangle are all direct, or else one involution is direct and the other two opposite.*

Proof. Let ABC be the self-polar triangle (fig. 68), and let PP' be a pair of points on the side BC and QQ' a pair on the side CA. Let R be the point of intersection of the lines PQ and AB, O that of AP' and BQ', and R' that of CO and AB. Then AP' is the polar of P, BQ' of Q, PQ of O, and CO of R. Hence R and R' are paired in the involution of conjugate points on AB. Let R'' be the point in which $P'Q'$ meets

If the involutions on BC and CA are direct, P and P' separate B and C, and Q and Q' separate C and A. It follows by Theorem 19, Chap. II, that R and R'' do not separate B and A. Hence by Theorems 7 and 8, Chap. II, R' is separated from R by A and B, and hence the involution on the line AB is direct.

On the other hand, if the involutions on BC and CA are not direct, P and P' do not separate B and C, and Q and Q' do not separate C and A. Hence R and R'' do not, and therefore R and R' do, separate A and B. Hence again the third involution is direct.

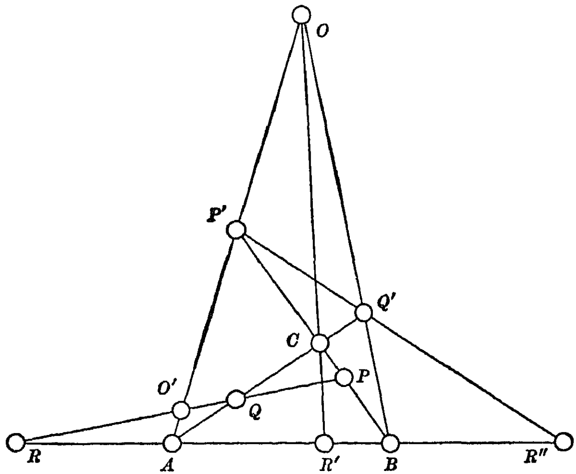


FIG. 68

We have thus shown that at least one of the three involutions is direct; and that if two are direct, so is also the third. From this the statement in the theorem follows.

The reasoning above is valid in any ordered projective space. Specializing to the real space, we have

COROLLARY 1. *The involutions on the sides of a self-polar triangle of a projective polar system in a real plane are all three elliptic, or else two are hyperbolic and the third is elliptic.*

THEOREM 37. *If the involutions of conjugate points on the sides of one self-polar triangle of a projective polar system in an ordered plane are direct, the involution of conjugate points on any line is direct.*

Proof. Let the given self-polar triangle on the sides of which the involutions of conjugate points are direct be ABC. The theorem will

BC in a point M which has a conjugate point N on BC. By the proposition which we are supposing proved, the involutions on the sides of the self-polar triangle, AMN, are direct; and by a second application of the same proposition, the involution of conjugate points on l is direct. Thus the proof of the theorem reduces to the proof that the involution of conjugate points on any line through A is direct.

Let such a line meet BC in a point P', and let P be the conjugate of P' in the involution on BC. Let Q and Q' be a conjugate pair distinct from A and C on the line AC, and let O, R, R', R'' have the same meaning as in the proof of the last theorem (fig. 68). Also let O' be the conjugate of O on the line AP', i.e. let O' be the intersection of AP' with PQ. Applying Theorem 19, Chap. II, to the triangle ABP' and the lines $O'R$ and OR', it follows that, since C and P do not separate B and P', and R and R' do separate A and B, O and O' are separated by A and P'. Hence the involution of conjugate points on the line AP' is direct.

Corollary 1. *If the involutions on two sides of a self-polar triangle of a polar system in an ordered plane are opposite, then two of the involutions on the sides of any self-polar triangle are opposite and the third is direct.*

Proof. If there were any self-polar triangle not satisfying the conclusion of the theorem, this would, by Theorem 36, be one for which all three involutions were direct. By Theorem 37 it would follow that the involutions on all lines were direct, contrary to hypothesis.

The propositions stated in the last two theorems and in the last corollary may evidently be condensed into the following:

Corollary 2. *Any projective polar system in an ordered plane is either such that the involution of conjugate points on any line is direct, or such that on the sides of any self-polar triangle two of the involutions are opposite and the third direct.*

Applying this result in a real plane, we have that every projective polar system is either such that all involutions of conjugate points are elliptic, or such that on the sides of any self-polar triangle two involutions are hyperbolic and the third elliptic. In the latter case let ABC be a self-polar triangle, AB and AC being the sides upon

which the involutions are hyperbolic. Let the double points of the involution on AB be C_1 and C_2, and those of the involution on AC be B_1 and B_2. The polar of C_1 is then the line C_1C. The conic section K^2 through C_1, C_2, B_1, B_2 and tangent to the line C_1C at C_1 has a polar system in which ABC is a self-polar triangle, and in which the given involutions are involutions of conjugate points. By § 93, Vol. I, these conditions are sufficient to determine a polarity. Hence the given polarity is the polar system of K^2. Thus we have

THEOREM 38. DEFINITION. *A projective polar system in a real plane is either the polar system of a real conic, or such that the involution of conjugate points on any line is elliptic. A polar system of the latter type is said to be* elliptic.

The existence of elliptic polar systems is easily seen as follows: Let ABC be any triangle, O any point not on a side of this triangle, P' the point of intersection of OA with BC, Q' the point of intersection of OB with CA, and P and Q any two points separated from P' and Q' by the pairs BC and CA respectively. By the theorems in § 93, Vol. I, there exists a polar system in which the triangle ABC is self-polar and the point O is the pole of the line PQ, and by the theorems in the present section this polar system is elliptic.

CHAPTER VI

INVERSION GEOMETRY AND RELATED TOPICS*

90. Vectors and complex numbers. The properties of the addition of vectors have been derived in § 42 from those of the group of translations. If the operation of multiplication is to satisfy the distributive law,

$$a(b+c) = ab + ac,$$

multiplication by a vector, a, must effect a transformation on the vector field such that $b+c$ is carried into the vector which is the sum of those to which b and c are carried. Since the group of translations is a self-conjugate subgroup of the Euclidean group, any similarity transformation of the vector field satisfies this condition.

Let us then consider the transformations effected on a vector field by the Euclidean group. Any similarity transformation is a product of a translation by a similarity transformation leaving an arbitrary point O invariant. But a translation carries every vector into itself. Hence any similarity transformation has the same effect on the field of vectors as a similarity transformation leaving O invariant. Hence the totality of transformations effected on the vector field by the Euclidean group is identical with the totality of transformations effected on it by the similarity transformations leaving O invariant. Since no such transformation changes every vector into itself, any two of them effect different transformations of the field of vectors. Hence we have

THEOREM 1. *The group of transformations effected by the Euclidean group in a plane upon the field of vectors is isomorphic with the group of similarity transformations leaving an arbitrary point invariant.*

To obtain a definition of multiplication we restrict attention to the

one transformation of this group carrying the points O and A to O and B respectively.

DEFINITION. Relative to an arbitrary vector OA, which is called the *unit vector*, the *product* of two vectors OX (where $X \neq O$) and OY is the vector OZ to which OY is carried by the direct similarity transformation carrying OA to OX, and is denoted by $OX \cdot OY$. In case $X = O$, $OX \cdot OY$ denotes the zero vector.

As obvious corollaries of this definition we have the following two theorems:

THEOREM 2. *The triad of points OAY is directly similar to the triad OXZ if and only if*

$$OZ = OX \cdot OY.$$

THEOREM 3. *The equation*

$$OZ = OX \cdot OY$$

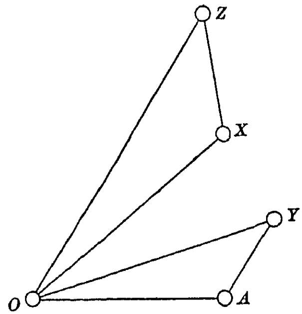


FIG. 69

is satisfied if and only if $\measuredangle AOX + \measuredangle AOY = \measuredangle AOZ$ *and* $Dist\,(OZ) = Dist\,(OX) \cdot Dist\,(OY)$, *the unit of distance being OA.*

Since the direct similarity transformations leaving a point O invariant form a group, the operation of multiplication must be associative, i.e.

$$OX \cdot (OY \cdot OZ) = (OX \cdot OY) \cdot OZ,$$

and also such that there is a unique inverse for every vector OB for which $O \neq B$, i.e. there must be a vector OY such that

$$OB \cdot OY = OA.$$

The group of direct similarity transformations leaving O invariant is commutative because it consists of the rotations about O (which form a commutative group by § 58) combined with dilations with O as center. Hence the operation of multiplication is commutative, i.e.

$$OX \cdot OY = OY \cdot OX.$$

The fact that the group of translations is self-conjugate under the group of displacements translates into the distributive law,

$$OX \cdot (OY + OZ) = OX \cdot OY + OX \cdot OZ.$$

Recalling the definition of a number system given in Chap. VI, Vol. I, we may summarize these results by saying

THEOREM 4. *With respect to the operation of addition described in § 42 and of multiplication defined in this section, a planar vector field is a commutative number system.*

In proving this theorem we have made use of no properties of the Euclidean group except such as hold for any parabolic metric geometry for which the absolute involution is elliptic. In case the absolute involution were hyperbolic, exceptions would have to be made corresponding to properties of the minimal lines.

The definition of multiplication of vectors as given here does not conflict with the notion of the ratio of collinear vectors as developed in Chap. III. For the quotient of two collinear vectors is a vector collinear with the unit vector OA, and the system of vectors collinear with OA constitutes a number system isomorphic with the real number system. Thus, if we denote the unit vector by **1**, any vector OX collinear with it may be denoted by

$$x\mathbf{1},$$

where, according to the definition of § 43, x is a real number and where, according to our present definition, x denotes OX itself.

Let us denote a vector OB such that the line OB is perpendicular to the line OA and such that $\text{Dist}\,(OB) = \text{Dist}\,(OA)$, by $\boldsymbol{i}$. Then by the definition of multiplication,

$$\boldsymbol{i}^2 = -\mathbf{1}.$$

Any vector collinear with $\boldsymbol{i}$ is expressible in the form $x\boldsymbol{i}$, where x is a vector parallel to **1**, and by Theorem 8, Chap. III, any vector whatever is expressible uniquely in the form

$$a\mathbf{1} + b\boldsymbol{i}.$$

The product of two vectors may be reduced by the associative, distributive, and commutative laws as follows:

$$\begin{aligned}(a\mathbf{1} + b\boldsymbol{i})(c\mathbf{1} + d\boldsymbol{i}) &= (a\mathbf{1} + b\boldsymbol{i})\,c\mathbf{1} + (a\mathbf{1} + b\boldsymbol{i})\,d\boldsymbol{i}\\ &= (ac - bd)\,\mathbf{1} + (bc + ad)\,\boldsymbol{i}.\end{aligned}$$

By comparison with §§ 3 and 14 this shows that

THEOREM 5. *A planar field of vectors is a number system isomorphic with the complex number system, i.e. the geometric number system of a complex line.*

The isomorphism in question is that by which the complex number $a + bi$ corresponds to the vector $a1 + bi$. Supposing that the fundamental points of the scale on the complex line are P_0, P_1, P_∞, this means that there is a correspondence between the complex line and the Euclidean plane in which P_0 corresponds to O, P_1 to A, and every point whose coördinate relative to the scale P_0, P_1, P_∞ is

$$a + bi$$

corresponds to the point Q of the Euclidean plane such that

$$OQ = a1 + bi.$$

One obvious property of this correspondence which we shall have to use later is that the points of the complex line which have real coördinates relative to the scale P_0, P_1, P_∞ correspond to the points of the line OA, or, in other words, that *the points of the chain* * $\mathsf{C}(P_0P_1P_\infty)$, *other than P_∞, correspond to the points on the real line OA.*

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The last theorem may therefore be stated in the following form:

THEOREM 6. *Let Q_0, Q_1, Q_∞ be three arbitrary points of a complex projective line l, and let P_0 and P_1 be two arbitrary points of a Euclidean plane π in whose line at infinity l_∞ an elliptic absolute involution is given. There exists a one-to-one and reciprocal correspondence Γ in which P_0 corresponds to Q_0, P_1 to Q_1, l_∞ to Q_∞, and every ordinary point of π to a point of l distinct from Q_∞. This correspondence is such that to every projective transformation of l leaving Q_∞ invariant, i.e. to every transformation of the form*

$$(1) \qquad x' = ax + b, \qquad a \neq 0,$$

there corresponds a direct similarity transformation of π, and conversely.

The question immediately arises, What group of transformations of π corresponds to the general projective group on l, i.e. to the set of transformations

$$(2) \qquad x' = \frac{ax+b}{cx+d}, \qquad \begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0\,?$$

The transformation of π corresponding to

$$(3) \qquad x' = 1/x$$

must change any point P to a point P' such that

$$P_0P' \quad P_0P = P_0P_1.$$

Hence, by Theorem 3, $\measuredangle PP_0P_1$ is congruent to $\measuredangle P_1P_0P'$. Therefore the orthogonal line reflection with P_0P_1 as axis must carry P to a point P'' of the line P_0P'. If P be regarded as a variable point of a line through P_0, it follows that the correspondence between P' and P'' is projective. In this correspondence P_0 corresponds to the point at infinity of the line P_0P', and each of the points in which this line meets the circle through P_1 with P_0 as center corresponds to itself. Hence the correspondence between P' and P'' on a given line through P_0 is an involution, and P' and P'' are conjugate points with respect to the circle. Hence (§ 71), if P be a variable point of the plane, the correspondence between P' and P'' is an inversion. Hence

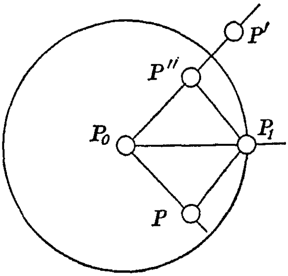

FIG. 70

the transformation of π corresponding to $x' = 1/x$ is the product of the orthogonal line reflection with P_0P_1 as axis and the inversion with respect to the circle through P_1 with P_0 as center.

Now any transformation (2) is evidently (cf. § 54, Vol. I) a product of transformations of the forms (1) and (3). But the transformation (1) has been seen to correspond to a direct similarity transformation, i.e. to a product of a dilation and a displacement. A displacement has been proved in Chap. IV to be a product of two orthogonal line reflections; and a dilation will now be shown to be a product of two or four inversions and orthogonal line reflections.

For consider a dilation Δ with a point O as center and carrying a point A to a point B. If O is not between A and B, there exists (Theorem 8, Chap. V) a pair of points C_1C_2 which separate A and B harmonically and have O as mid-point. Let I_1 be the inversion with respect to the circle with O as center and passing through C_1. The transformation $\mathrm{I}_1\Delta$ leaves invariant all points of the circle through A with O as center, and effects a projectivity on each line through O which interchanges O and the point at infinity. The projectivity on each line through O is therefore the involution carrying each point to a conjugate point with regard to the circle through A with O as center. Hence $\mathrm{I}_1\Delta$ is an inversion, I_2, with respect to this circle. From $\mathrm{I}_1\Delta = \mathrm{I}_2$ follows $\Delta = \mathrm{I}_1\mathrm{I}_2$. If O is between A and B, let Λ be the point reflection with O as center. The product $\Lambda\Delta$ is a dilation such that O is not between A and $\Lambda\Delta(A)$. Hence $\Lambda\Delta$ is a product of two inversions I_1, I_2 and $\Delta = \Lambda\mathrm{I}_2\mathrm{I}_1$. Since Λ is a product of two orthogonal line reflections, Δ is a product of four inversions and orthogonal line reflections.

Hence *any projective transformation of a complex line l corresponds under Γ to a transformation of a real Euclidean plane π which is a product of an even number of inversions and orthogonal line reflections.*

The converse of this proposition is also valid. In order to prove it we need only verify (α) that the product of two orthogonal line reflections in π corresponds to a projectivity of l, (β) that the product of an orthogonal line reflection Λ and an inversion P of π corresponds to a projectivity of l, and (γ) that the product of two inversions $\mathrm{P}_1\mathrm{P}_2$ of π corresponds to a projectivity of l. The first of these statements is a

To prove (β) let us first consider the case where the axis of Λ passes through the center O of P. Let O_1 be one of the points in which the axis of Λ meets the invariant circle of P, X be any point of π, and $X' = \Lambda\mathrm{P}(X)$. The considerations given above in connection with the transformation (3) show that

$$OX' = \frac{OO_1}{OX},$$

and hence that $\Lambda\mathrm{P}$ corresponds to a transformation of l of the same type as (3), i.e. to an involution. Moreover, $\Lambda\mathrm{P}$ is obviously the same as $\mathrm{P}\Lambda$. In case the axis of Λ does not pass through the center of P, let Λ' be an orthogonal line reflection whose axis passes through the center of P. Then

$$\Lambda\mathrm{P} = \Lambda\Lambda' \cdot \Lambda'\mathrm{P} \quad \text{and} \quad \mathrm{P}\Lambda = \mathrm{P}\Lambda' \cdot \Lambda'\Lambda.$$

The products $\Lambda\Lambda'$ and $\Lambda'\Lambda$ correspond to projectivities by Theorem 6, and $\mathrm{P}\Lambda' = \Lambda'\mathrm{P}$ corresponds to an involution by what has just been proved. Hence $\Lambda\mathrm{P}$ and $\mathrm{P}\Lambda$ correspond to projectivities.

To prove (γ) let Λ be an orthogonal line reflection whose axis contains the centers of P_1 and P_2. Then

$$\mathrm{P}_1\mathrm{P}_2 = \mathrm{P}_1\Lambda \cdot \Lambda\mathrm{P}_2.$$

The products $\mathrm{P}_1\Lambda$ and $\Lambda\mathrm{P}_2$ correspond to projectivities by (β). Hence $\mathrm{P}_1\mathrm{P}_2$ corresponds to a projectivity. Thus we have the important result:

THEOREM 7. *A projective transformation on a complex line corresponds under* Γ *to a transformation of the real Euclidean plane which is a product of an even number of inversions and orthogonal line reflections, and, conversely, any transformation of the real Euclidean plane of this type corresponds to a projectivity of the complex line.*

92. The inversion group in the real Euclidean plane.

DEFINITION. The transformations of a Euclidean plane and its line at infinity which are products of orthogonal line reflections and inversions are called *circular transformations,* and any circular transformation which is a product of an even number of inversions and orthogonal line reflections is said to be *direct.*

THEOREM 8. DEFINITION. *The set of all circular transformations*

involution is given constitute a group which is called the inversion group. *The set of direct circular transformations form a subgroup of the inversion group, which, if the Euclidean plane is real, is isomorphic with the projective group of a complex line.*

The first part of this theorem is an obvious consequence of the definition, and the second is equivalent to Theorem 7. That not all circular transformations are direct is shown by the special case of an inversion. An inversion is not a direct circular transformation, because it leaves invariant all points of a circle and hence cannot correspond under Γ to a projectivi'y. Combining Theorems 8 and 6 we have

COROLLARY. *In a real Euclidean plane the group of circular transformations leaving l_∞ invariant is the Euclidean group, and the direct circular transformations leaving l_∞ invariant are the direct similarity transformations.*

The isomorphism between the group of direct circular transformations and the projective group on the line may be used as a source of theorems about the former. Thus the fundamental theorem of projective geometry (Assumption P) translates into the following theorem about the real Euclidean plane:

THEOREM 9. *A direct circular transformation which leaves three ordinary points, or two ordinary points and l_∞, invariant is the identity. There exists a direct circular transformation carrying any three distinct ordinary points A, B, C respectively into three distinct points A', B', C' respectively, or into A', B', and l_∞ respectively.*

Now consider a circular transformation Π which is not direct and which leaves three distinct points A, B, C invariant. By definition

$$\Pi = \Lambda_{2n+1} \cdot \Lambda_{2n} \cdots \Lambda_2 \cdot \Lambda_1,$$

where Λ_i $(i = 1, 2, \cdots, 2n+1)$ is an inversion or an orthogonal line reflection. Let Λ be an orthogonal line reflection whose axis contains A, B, C, if these points are collinear, or an inversion with respect to the circle containing them in case they are not collinear. Then $\Lambda\Pi$ is a direct circular transformation leaving A, B, C invariant. Hence

$$\Lambda\Pi = 1.$$

Since Λ is of period two, this implies

The same argument applies in case one of the points A, B, C is replaced by l_∞. Hence we have

THEOREM 10. *A circular transformation which is not direct and leaves invariant three distinct ordinary points A, B, C, or two ordinary points A, B, and l_∞, is an orthogonal line reflection or an inversion according as the invariant points are collinear or not.*

THEOREM 11. *If Π is a circular transformation and Λ an inversion or orthogonal line reflection, $\Pi\Lambda\Pi^{-1}$ is an inversion or orthogonal line reflection.*

Proof. Let A, B, C be three of the invariant points of Λ; then $\Pi\Lambda\Pi^{-1}$ leaves $\Pi(A)$, $\Pi(B)$, $\Pi(C)$ invariant. If

$$\Pi = \Lambda_1\Lambda_2 \cdots \Lambda_n,$$

where $\Lambda_1, \cdots, \Lambda_n$ are orthogonal line reflections or inversions, then

$$\Pi\Lambda\Pi^{-1} = \Lambda_1\Lambda_2 \cdots \Lambda_n\Lambda\Lambda_n \cdots \Lambda_2\Lambda_1,$$

and is thus a product of an odd number of orthogonal line reflections or inversions. Hence by the last theorem it is an orthogonal line reflection or an inversion.

The invariant elements of $\Pi\Lambda\Pi^{-1}$ are those to which the invariant elements of Λ are carried by Π. Since $\Pi\Lambda\Pi^{-1}$ is an inversion or an orthogonal line reflection, we have

COROLLARY 1. *Any circular transformation carries any circle into a circle or into the set of points on an ordinary line and on l_∞. It carries the set of points on l_∞ and an ordinary line into a set of this sort or into a circle.*

COROLLARY 2. *If C^2 and K^2 are any two circles and l any line, there exists a direct circular transformation carrying C^2 to K^2 and one carrying C^2 to the set of all points on l and l_∞.*

Proof. Let A, B, C be any three points of C^2, let A', B', C' be any three points of K^2, and let A', B' be any two points of l. By Theorem 9, there exist direct circular transformations Π and Π' such that

$$\Pi(ABC) = A'B'C' \quad \text{and} \quad \Pi'(ABC) = A'B'l_\infty.$$

Since A', B', C' are not collinear, the set of points into which Π carries C^2 must be a circle; and since there is only one circle containing A', B', C', this circle is K^2. Since there is no circle containing A', B', and l_∞, the set of points into which Π' carries C^2 must be the set of

points on l_∞ and an ordinary line. Since the ordinary line contains A' and B', it must be l.

An inversion (§ 71) transforms all lines through its center into themselves and interchanges the center with l_∞. Hence, by the last two corollaries, we have at once

COROLLARY 3. *An inversion carries a circle through its center into the set of points on l_∞ and a line not passing through the center.*

COROLLARY 4. *A pair of circles which touch each other is carried by an inversion into a pair of circles which touch each other, or into a circle and a tangent line together with l_∞, or into two parallel lines and l_∞.*

Proof. Let C^2 and K^2 be two circles which touch each other Since an inversion is a one-to-one reciprocal correspondence except for the origin and l_∞, if neither C^2 nor K^2 passes through the origin, they must be carried into two circles having only one point in common and which therefore touch each other. If C^2 passes through the origin and K^2 does not, C^2 is carried into l_∞ and an ordinary line l, while K^2 is carried into a circle K_1^2 which has one and only one point in common with the line pair $l_\infty l$. Since l_∞ cannot meet K_1^2 in a real point, l meets it in a single point and therefore is tangent. If C^2 and K^2 both pass through the center of inversion, they are transformed into l_∞ and a pair of ordinary lines l, m. Since C^2 and K^2 have only the center of inversion in common and this is transformed into l_∞, the lines l and m can have no ordinary point in common. Hence l and m are parallel.

It was remarked in § 90 (just before the fine print at the end) that the correspondence Γ between the complex line and the real Euclidean plane is such that the points of a certain chain $\mathsf{C}(P_0P_1P_\infty)$, with the exception of P_∞, correspond to the points of a certain Euclidean line l. Since P_∞ corresponds to l_∞, the chain $\mathsf{C}(P_0P_1P_\infty)$ corresponds to the line pair ll_∞. Under the projective group on a line any two chains are equivalent; and under the group of direct circular transformations any circle is equivalent to any circle or any line pair ll_∞ (Cor. 2). Hence we have

THEOREM 12. *The correspondence Γ is such that chains in the complex line correspond to real circles or to line pairs ll_∞, where l is*

The theory of chains on a complex line is therefore equivalent to the theory of the real circles and lines of a Euclidean plane. In view of this equivalence we shall freely transform the terminology of the complex line to the Euclidean plane, and vice versa. Thus we shall speak of the cross ratio of four points in the Euclidean plane and of pencils of chains in the complex line. The exercises below contain a number of important theorems some of which can be obtained directly from the definitions in § 71 and some of which can be proved most simply by translating projective theorems on the complex line into the terminology of the Euclidean plane.

DEFINITION. An *imaginary circle* is an imaginary conic through the circular points such that its polar system transforms real points into real lines.

The definition of an inversion given in § 71 applies without change to the case of imaginary circles.

On the geometry of circles in general the reader is referred to the papers by Möbius in Vol. II of his collected works; to those by Steiner in Vol. I (especially pp. 16–83, 461–527) of his collected works; to Vol. II, Chaps. II, III, of the textbook by Doehlemann referred to in Ex. 4; and to the forthcoming book by J. L. Coolidge, A Treatise on the Circle and the Sphere, Oxford, 1916.

EXERCISES

1. An inversion with respect to an imaginary circle is a product of an inversion with respect to a real circle and a point reflection having the same center as the circle.

2. The inverse points on any line through the center O of a circle C^2 are the pairs of an involution having O as center. If A_1 and A_2 are any two inverse points, $OA_1 \cdot OA_2$ is a constant, which in case of a real circle is equal to $(OC)^2$, C being a point of C^2.

3. Two pairs of points AA' and BB' are inverse with respect to a circle with O as center if and only if (1) O is collinear with the pairs AA' and BB', and (2) the ordered triads OAB and $OB'A'$ are similar, but not directly similar.

4. A linkage which consists of a set of six bars OA, OC, AB, BC, CD, DA, jointed movably at the points O, A, B, C, D, and such that Dist $(OA) =$ Dist (OC) and $ABCD$ is a rhombus, is called a "Peaucellier inversor." If O is held fixed and B varies, the locus of D is inverse to that of B with respect to a circle with O as center. If B be constrained, say by an additional link, to move on a circle through O, D describes a line. On the gene

subject of linkages, cf. K. Doehlemann, Geometrische Transformationen, Vol. II, p. 90, Leipzig, 1908, and A. Emch, Projective Geometry, §§ 62–67, New York, 1905.

5. If A, B, C, D are four points of a Euclidean plane,

$$\mathrm{R}\,(AB,\,CD) = ke^{i\theta},$$

where
$$k = \frac{\text{Dist}\,(AC)}{\text{Dist}\,(AD)} \div \frac{\text{Dist}\,(BC)}{\text{Dist}\,(BD)} \quad \text{and} \quad \theta = \alpha - \beta,$$

where α and β are the measures of $\measuredangle\, CAD$ and $\measuredangle\, CBD$ respectively. The number k is invariant under the inversion group, and θ under the group of direct circular transformations. The four points are on a circle or collinear if $\theta = 0$.

6. Construct a point having with three given points a given cross ratio.

7. If Π is any circular transformation, the points $O = \Pi^{-1}(l_\infty)$ and $O' = \Pi(l_\infty)$ are called its vanishing points. The lines through O are transformed by Π into the lines through O'. If X is any point of the plane, and $X' = \Pi(X)$, then $\text{Dist}\,(OX) \cdot \text{Dist}\,(O'X')$ is a constant, called the *power* of the transformation (cf. § 43).

8. Let A and B be two points not collinear with O and let $\Pi(A) = A'$, $\Pi(B) = B'$. The ordered point triads OAB and $O'B'A'$ are directly similar if Π is direct, and similar, but not directly so, if Π is not direct.

9. The equations of an inversion relative to rectangular nonhomogeneous coördinates, having the center of inversion as origin, are

$$x' = \frac{kx}{x^2 + y^2}, \qquad y' = \frac{ky}{x^2 + y^2}.$$

The circle of inversion is real or imaginary according as $k > 0$ or $k < 0$.

10. The coördinate system for the real Euclidean plane obtained by means of the isomorphism of the Euclidean group with the projective group leaving a point invariant on a complex line is such that the coördinate z of any point is $x + iy$, where x and y are the coördinates in a system of rectangular nonhomogeneous coördinates and $i^2 = -1$. The points z of a circle satisfy the condition

$$z = \frac{at + b}{ct + d}, \qquad \begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0,$$

where t is real and variable and a, b, c, d are complex and fixed. If $c = 0$, this circle reduces to a line.

11. The circles orthogonal to $z = \dfrac{at + b}{ct + d}$ are

$$z = \frac{(a + b\beta)\, it + b + a\alpha}{(c + d\beta)\, it + d + c\alpha},$$

where α and β are real.

12. The circles through two points z_1, z_2 are given by

$$z = \frac{atz_1 + z_2}{at + 1}.$$

13. A circle with z_1 as center is given by

$$z - z_1 = ke^{i\theta},$$

where $0 \equiv \theta < 2\pi$ and k is a real constant.

14. The centers of the circles circumscribing the four triangles formed by the sides of a complete quadrilateral are on a circle. This circle is called the *center circle* of the complete quadrilateral. The centers of the center circles of the five complete quadrilaterals formed by the sides of a complete five-line are on a circle called the *center circle* of the five-line. Generalize this result.

93. Generalization by inversion. By the corollary of Theorem 8 the set of direct circular transformations leaving l_∞ invariant is the group of direct similarity transformations, and the set of all circular transformations leaving l_∞ invariant is the Euclidean group. This is the basis of a method of *generalization by inversion* entirely analogous to the *generalization by projection* employed in § 73.

In case a figure F_1 which is under investigation can be transformed by one or more inversions into a known figure F_2, then such of the relations among the elements of F_2 as are invariant under circular transformations must hold good among the corresponding elements of F_1.

In order to apply this method it is necessary to know relations which are left invariant by the circular transformations. The most elementary of these are given in the last section, but perhaps the most important property of an inversion for this purpose is that of *isogonality*, or "preservation of angles."

DEFINITION. If C_1^2 and C_2^2 are two circles having a point Q in common, and m_1 and m_2 are the tangents to C_1^2 and C_2^2 respectively at Q, the measure (according to § 72) of the ordered line pair m_1m_2 is called the *angular measure* of the ordered pair of circles at Q, or simply the *angle* between the two circles at Q. If C_1^2 is any circle, m_2 a line meeting it in a point Q, and m_1 the tangent to C_1^2 at Q, the measure of the ordered line pair m_2m_1 is called the *angle* between m_2 and C_1^2, and the measure of m_1m_2 is called the *angle* between C_1^2 and m_2. The measure of a line pair m_1m_2 is called the *angle** between m_1 and m_2.

THEOREM 13. *An angle a between two circles or a circle and a line or between two lines is changed into $\pi - a$ by an inversion or*

* In accordance with common usage, we are here using the term "angle" to denote a number, in spite of the fact that we use it in § 28 to denote a geometrical

an orthogonal line reflection and is left unaltered by any direct circular transformation.

Proof. The statement with regard to direct circular transformations is an obvious consequence of the one with regard to inversions and orthogonal line reflections. What we have to prove is, therefore, the following:

Let Π be an inversion or an orthogonal line reflection, and let l_1 and l_2 be two lines meeting in a point P such that $\Pi(P) = Q$ is an ordinary point. If l_1 is carried by Π into a line, let this line be denoted by m_1; and if l_1 (together with l_∞) is carried to a circle C_1^2, let m_1 denote the tangent to C_1^2 at Q; likewise, if l_2 is carried by Π into a line, let this line be denoted by m_2; and if l_2 (together with l_∞) is carried to a circle C_2^2, let m_2 denote the tangent to C_2^2 at Q. The two ordered pairs of lines l_1l_2 and m_1m_2 are symmetric.

In case Π is an orthogonal line reflection, $m_1 = \Pi(l_1)$ and $m_2 = \Pi(l_2)$, and the proposition is a direct consequence of the definition of the term "symmetric" (§ 57). Suppose, then, that Π is an inversion having a point O as center.

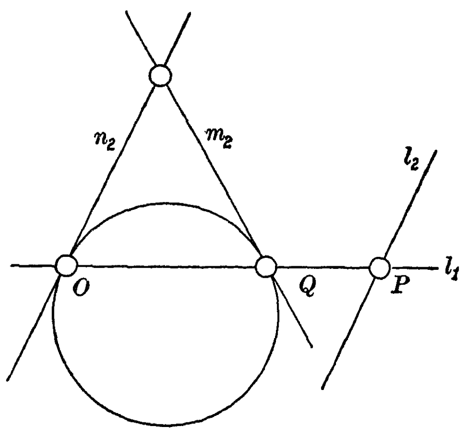

FIG. 71

One of the lines l_1, l_2, say l_1, can be transformed into itself if and only if l_1 is on O. By hypothesis $O \neq P$; hence if $\Pi(l_1) = l_1$, the line l_2 goes into the set of points different from O on a circle C_2^2 through O and Q. Then m_2 is the tangent to C_2^2 at Q. Any line through O which meets l_2 in an ordinary point X meets C_2^2 in the point which corresponds to X under the inversion. Hence the line n_2 through O and tangent to C_2^2 cannot meet l_2 in an ordinary point, and is therefore parallel to l_2. Hence the line pair l_1l_2 is congruent to the pair l_1n_2. The line m_2 is the tangent to C_2^2 at Q. Since l_1n_2 is carried to l_1m_2 by the orthogonal line reflection whose axis is the perpendicular bisector of OQ, the pair

If neither of the lines l_1, l_2 is transformed into itself, neither passes rough O. Let l denote the line OP. Then by the last paragraph ll_1 symmetric with lm_1, and ll_2 with lm_2. But by Theorem 13, Chap. IV, e symmetry which carries ll_1 to lm_1 must be identical with that iich carries ll_2 to lm_2. Hence l_1l_2 is symmetric with m_1m_2.

As an exercise in generalization by inversion let us prove the following:

THEOREM 14. *If three circles C_1^2, C_2^2, C_3^2 meet in a point O in such a way that :h pair of them makes an angle $\frac{\pi}{3}$, and also meet by pairs in three other points Q, R, the circle (or line) through P, Q and R makes with each of the other cles an angle $\frac{\pi}{3}$.*

Proof. The pair of circles which meet at O obviously make the angle $\frac{\pi}{3}$ at :h of the points P, Q, R. An inversion Π with respect to a circle having O center must therefore change them into the sides of an equilateral triangle. .e circle circumscribing this triangle makes the angle $\frac{\pi}{3}$ with each of the .es. But since this circle is the transform of the circle PQR by Π, the nclusion of the theorem follows.

As a second application of the theory of inversion, in combination with jective methods, we may consider the theorem of Feuerbach on the nine .nt circle (cf. Ex. 2, § 73).

THEOREM 15. *The nine-point circle of a triangle touches the four inscribed circles.*

Proof. Let the given triangle be ABC, and let the mid-points of the pairs C, CA, AB be A_1, B_1, C_1 respectively. The nine-point circle is the circle ntaining A_1, B_1, C_1.

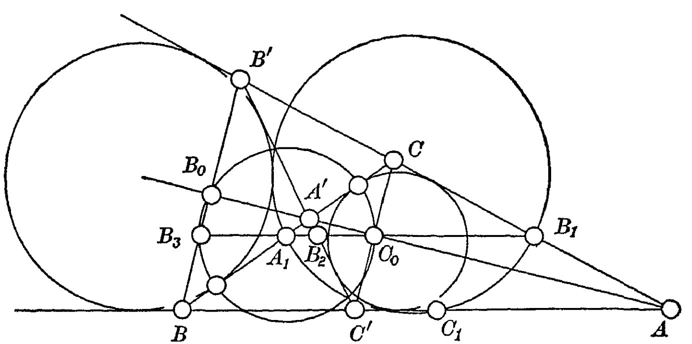

FIG. 72

is one line, l, besides AB, BC, CA, which touches both K_1^2 and K_2^2. Let $A'B'C'$ be the points in which l meets the sides BC, CA, AB respectively. Then AA', BB', CC' are the pairs of opposite vertices of a complete quadrilateral circumscribing both K_1^2 and K_2^2, and the diagonal triangle of this quadrilateral is a self-polar triangle both for K_1^2 and K_2^2 (§ 44, Vol. I). Since the side AA' of this triangle is the line of centers of K_1^2 and K_2^2, the other two sides, BB' and CC', are parallel to each other and perpendicular to AA'. Let their points of intersection with AA' be B_0 and C_0 respectively. These two points are conjugate with respect to both circles, and hence must be the limiting points of the pencil of circles containing K_1^2 and K_2^2. The radical axis of the pencil of circles is the perpendicular bisector of the pair B_0C_0, and hence (§ 40) passes through the mid-points of all the pairs BC, $B'C'$, BC', $B'C$, B_0C_0. In particular the radical axis of K_1^2 and K_2^2 passes through A_1, the mid-point of BC. Hence there is a circle G^2 with A_1 as center and passing through B_0 and C_0.

Let Γ be the inversion with respect to G^2. Since this circle passes through B_0 and C_0, it is orthogonal both to K_1^2 and K_2^2 (Theorem 34, § 71), and hence Γ transforms each of these circles into itself. We shall now prove that Γ transforms l into the nine-point circle.

Let B_2 be the point in which A_1B_1 meets l. Since A_1B_1 is parallel to AB, it is not parallel to l, and hence B_2 is an ordinary point. Since A_1B_1 contains the mid-point A_1 of the pair CB and is parallel to BC', it contains the mid-point C_0 of the pair CC'. The involution which Γ effects on the line A_1B_1 must have C_0 as one of its double points and A_1 as its center; hence the other double point must be the point B_3 in which A_1B_1 meets BB', because A_1 is the mid-point of the pair C_0B_3. Thus G^2 passes through B_3 as well as through C_0. But since

$$B_0A'C_0A \overset{B'}{\underset{\wedge}{=}} B_3B_2C_0B_1,$$

B_1 and B_2 are harmonically conjugate with respect to C_0 and B_3. Hence Γ transforms B_2 to B_1.

In like manner it can be shown that if C_2 is the point in which A_1C_1 meets l, Γ transforms C_2 to C_1. Since any line whatever is transformed by Γ to a circle through A_1, it follows that l is transformed to the circle through A_1, B_1, and C_1, i.e. to the nine-point circle. By Theorem 11, Cor. 4, since l is tangent to K_1^2 and K_2^2, the nine-point circle touches K_1^2 and K_2^2. Since it has not been specified which of the bisectors of $\measuredangle CAB$ contains the centers of K_1^2 and K_2^2, this argument shows that the nine-point circle touches all four inscribed circles.

EXERCISES

1. Any three points can be carried by an inversion into three collinear points.

2. Two nonintersecting circles can be carried by an inversion into concentric circles.

3. Any direct circular transformation is a product of an inversion and

4. A product of two inversions is an involution if and only if the circles are orthogonal.

5. Of four circles mutually perpendicular by pairs, three can be real.

6. The nine-point circle meets the circle through C_0 having A_1 as center in points of the line $A'B'$.

7. The nine-point circle of a triangle touches the sixteen circles inscribed to the triangle or to any of the triangles formed by pairs of its vertices with the orthocenter.

8. Let three circles C_1^2, C_2^2, C_3^2 meet in a point O, and let P_1, P_2, P_3 be the other points of intersection of the pairs $C_2^2C_3^2$, $C_3^2C_1^2$, $C_1^2C_2^2$ respectively. If Q_1 be any point of C_1^2, Q_2 the point of C_2^2 collinear with and distinct from Q_1 and P_3, and Q_3 the point of C_3^2 collinear with and distinct from Q_2 and P_1, then Q_3, P_2, and Q_1 are collinear.

9. *The problem of Apollonius.* Construct the circles touching three given circles. Cf. Pascal, Repertorium der Höheren Mathematatik, II 1, Chap. II, on this and the following exercise.

10. *The problem of Malfatti.* Given a triangle, determine three circles each of which is tangent to the other two and also to two sides of the triangle.

94. Inversions in the complex Euclidean plane. Thus far we have dealt only with a real Euclidean plane. The definition of an inversion given in § 71, however, applies without change in the complex Euclidean plane; i.e. two points A_1, A_2 are inverse with respect to a circle C^2, provided they are conjugate with respect to C^2 and collinear with its center. The transformation thus defined is obviously one to one and reciprocal for all points of the complex projective plane except those on the sides of the triangle OI_1I_2, where O is the center of C^2, and I_1 and I_2 are the circular points at infinity. Any point of l_∞ is carried to O by the inversion, and O is carried to every point of l_∞. The circular point I_1 is transformed to every point of the line OI_1, and every point of the line OI_1 is transformed to I_1. In like manner I_2 is transformed to every point of the line OI_2, and every point of this line is carried to I_2.

DEFINITION. The sides of the triangle OI_1I_2 are called the *singular lines* of the inversion with respect to C^2, and the points on these lines are called its *singular points*.

The principal properties of an inversion may be inferred from the following construction: If A_1 is any point not on a side of the triangle OI_1I_2, let B_1 and B_2 be the points distinct from I_1 and I_2 (fig. 73) in which the lines A_1I_1 and A_1I_2 respectively meet C^2. Let A_2 be the point of intersection of I_1B_2 and I_2B_1. The points A_1 and A_2

are mutually inverse because, by familiar theorems on conics, they are conjugate with regard to C^2 and collinear with O.

From this construction it is evident in the first place that all points, except I_1 of the line A_1I_1, are transformed into points of the line A_2I_2, and vice versa. Hence an inversion transforms the minimal lines through I_1 into the minimal lines through I_2, and vice versa. Moreover, the correspondence between the two pencils of minimal lines is such that if B is a variable point of C^2, the line I_1B always corresponds to I_2B. In other words, the correspondence effected by an inversion between the two pencils of minimal lines is a projectivity generating the invariant circle C^2.

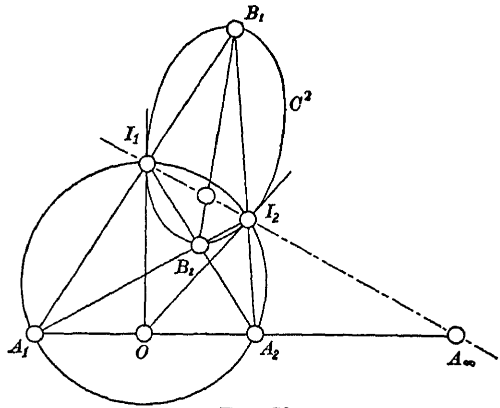

FIG. 73

The definitions of circular and of direct circular transformations, given in § 92, apply without change in the complex Euclidean plane. The result just obtained therefore implies that *any direct circular transformation transforms each pencil of minimal lines projectively into itself, and any nondirect circular transformation transforms each pencil of minimal lines projectively into the other.*

Now suppose that A_1 is a variable point on any line l not containing I_1 or I_2.

(4) $$I_1[A_1] \stackrel{l}{\overline{\overline{\wedge}}} I_2[A_1].$$

Since B_1 and B_2 are always on the conic C^2,

(5) $$I_1[A_1] \overline{\wedge} I_2[B_1],$$

and

(6) $$I_2[A_1] \overline{\wedge} I_1[B_2].$$

Hence

(7) $$I_2[B_1] \overline{\wedge} I_1[B_2].$$

But corresponding lines of these two pencils intersect in the vari-

or on a line. In the projectivity (5) the line I_2O corresponds to l_∞; in (4) l_∞ corresponds to itself; and in (6) l_∞ corresponds to I_1O. Hence in (7) the line I_2O corresponds to I_1O, and so the circle or line generated by (7) passes through O.

This result may be stated in a form which takes account of the singular elements, as follows: *Any degenerate conic consisting of l_∞ and a nonminimal line is carried by an inversion with respect to C^2 into a conic (degenerate or not) which passes through I_1, I_2, and O.*

Next suppose A_1 to be a variable point on any nondegenerate conic through I_1 and I_2. In this case

$$(8) \qquad I_1[A_1] \;\overline{\wedge}\; I_2[A_1],$$

and hence by the projectivities (5) and (6) we have

$$(9) \qquad I_2[A_2] \;\overline{\wedge}\; I_1[A_2].$$

Hence A_2 is again on a conic through I_1 and I_2, which can degenerate only if l_∞ corresponds to itself under (9). The latter case implies, by (5) and (6), that I_1O and I_2O correspond under (8) or, in other words, that the locus of A_1 passes through O. Hence *any nondegenerate conic K^2 through I_1 and I_2 corresponds by the inversion with respect to C^2 to a conic through I_1 and I_2, which degenerates into a pair of lines, one of which is l_∞, only in case K^2 passes through O.*

This result, together with the other statement italicized above, amount to an extension of Cors. 1 and 3 of Theorem 11 to the complex Euclidean plane. From our present point of view we can also establish the following theorem, which did not come out of the reasoning in § 92.

THEOREM 16. *The correspondence between two circles which are homologous under an inversion is projective.*

Proof. If A_1 is a variable point of one circle and A_2 of the other, then, in the notation above, $I_2B_1 = I_2A_2$, and hence by (5)

$$I_1[A_1] \;\overline{\wedge}\; I_2[A_2],$$

which is a necessary and sufficient condition that the correspondence

The same reasoning also applies in case one or both of the conics which are the loci of A_1 and A_2 degenerate. We thus have

COROLLARY. *A projective correspondence is established by an inversion between any two homologous lines or between a line and its homologous set of points on a circle.*

The proof of Theorem 13 on the preservation of angles under a circular transformation applies without change in the complex Euclidean plane. This theorem can also be proved by the use of considerations with regard to the circular points. We shall give the argument for the case of orthogonal circles, leaving it as an exercise for the reader to derive the proof along these lines for the general case.

It has been proved in § 71 that the circles through two points A_1, A_2 are orthogonal to the circles through two points B_1, B_2 if and only if the pairs A_1A_2, B_1B_2, and I_1I_2 are pairs of opposite vertices of a complete quadrilateral (cf. fig. 73). The sides I_1A_1, I_1A_2, I_2A_1, I_2A_2 of such a quadrilateral are transformed by an inversion relative to any circle into four lines through I_1 and I_2. Hence the points A_1, A_2, B_1, B_2 are transformed into four points A_1', A_2', B_1', B_2' such that I_1I_2, $A_1'A_2'$, and $B_1'B_2'$ are pairs of opposite vertices of a complete quadrilateral. Hence the pencils of circles through A_1, A_2 and B_1, B_2 respectively are transformed into two pencils such that the circles of one pencil are orthogonal to those of the other.

With this result it is easy to prove that Theorems 8–11, 13, and their corollaries hold in the complex Euclidean plane, proper exceptions being made so as to exclude minimal lines and pairs of points on minimal lines. This is left as an exercise.

95. Correspondence between the real Euclidean plane and a complex pencil of lines. The correspondence between a complex one-dimensional form and the points of a real Euclidean plane, together with l_∞, can be established in a particularly interesting way if the one-dimensional form be taken as the pencil of lines on one of the circular points of the line at infinity of the Euclidean plane.

line through I_1, except l_∞, contains one and only one real point of the Euclidean plane. Let us denote by Γ' the correspondence by which l_∞ corresponds to itself and the other lines through I_1 correspond each to the real point which it contains.

By § 94 a direct circular transformation transforms the pencil of lines on I_1 projectively into itself. Hence every direct circular transformation corresponds under Γ' to a projectivity of the lines on I_1.

By Theorem 9 there is one and only one direct circular transformation carrying an ordered triad of distinct points to an ordered triad of distinct points; and by Assumption P there is one and only one projectivity carrying an ordered triad of lines of a pencil to any ordered triad of the pencil. Hence a given projectivity of the pencil of lines on I_1 can correspond under Γ' to only one direct circular transformation. In other words, Γ' sets up a simple isomorphism between the projective group of a complex one-dimensional form and the group of direct circular transformations.

The correspondence between the points of a real line and the lines joining them to I_1 is evidently projective. Since the cross ratio of four points of a real line is real, so is the cross ratio of the lines joining them to I_1. Hence any real line together with l_∞ corresponds under Γ' to a chain. Since any two chains of a one-dimensional form are projectively equivalent, and any circle of the Euclidean plane is equivalent under the inversion group to an ordinary line and l_∞, it follows that under Γ' any chain corresponds to a circle and any circle to a chain.

The correspondence Γ' may be used to transfer the theory of involution from the complex pencil of lines to the Euclidean plane. Let AA', BB', CC' be pairs of opposite vertices of a complete quadrilateral of the Euclidean plane. The pairs of lines joining these point pairs to I_1 are pairs of an involution. Hence

THEOREM 17. *The pairs of opposite vertices of a complete quadrilateral are pairs of an involution, i.e. they are pairs of homologous points in a direct circular transformation of period two.*

In other words, the pairs of opposite vertices of a complete quadrilateral constitute the image under Γ' (and hence under Γ) of a quadrangular set. While the converse of this proposition is not true, the proposition can be generalized by inversion so as to give

a construction for the most general quadrangular set in which no four of the six points are on the same circle or line (cf. Ex. 1, below). We shall state the construction in terms of chains.*

THEOREM 18. *Given two pairs of points AA' and BB' and a point C such that no four of the five points are on the same chain. The chains $\mathsf{C}(AB'C)$ and $\mathsf{C}(A'BC)$ either meet in a point D other than C or touch each other at C. In the latter case let D denote C. The chains $\mathsf{C}(DAB)$ and $\mathsf{C}(DA'B')$ meet in a point C' such that AA', BB', CC' are pairs of an involution.*

Proof. Consider the figure in the Euclidean plane (together with l_∞) corresponding under Γ' to the figure described in the theorem. If $\Gamma'(D) \neq l_\infty$, $\Gamma'(D)$ can be transformed to l_∞ by an inversion I. Under $\mathrm{I}\Gamma'$ the four chains $\mathsf{C}(AB'C)$, $\mathsf{C}(A'BC)$, $\mathsf{C}(DAB)$, and $\mathsf{C}(DA'B')$ correspond to Euclidean lines (with l_∞), and hence AA', BB', CC' correspond to the vertices of a complete quadrilateral; so that the theorem reduces to Theorem 17. If $\Gamma'(D) = l_\infty$, the theorem reduces directly to Theorem 17.

COROLLARY. *Three pairs of points on a complex line AA', BB', CC', such that the chains $\mathsf{C}(A'B'C')$, $\mathsf{C}(A'BC)$, $\mathsf{C}(AB'C)$, $\mathsf{C}(ABC')$ are distinct, are pairs of an involution if and only if the four chains have a point in common.*

EXERCISES

1. Three pairs of points of the same chain AA', BB', CC' are in involution if for any point D not in the chain the chains $\mathsf{C}(DAA')$, $\mathsf{C}(DBB')$, $\mathsf{C}(DCC')$ are in the same pencil.

2. Derive Ex. 15, § 81, from the theory of involutions in a plane.

3. If AA', BB', CC' are pairs of opposite vertices of a complete quadrilateral, the three circles having AA', BB', CC' respectively as ends of their diameters belong to the same pencil, and the radical axis of this pencil passes through the center of the circle circumscribing the diagonal triangle of the quadrilateral.

4. Construct the double points of an involution in a Euclidean plane with ruler and compass.

* This puts in evidence the fact that while the geometry of real one-dimensional forms depends essentially on constructions implying the existence of two-dimen-

96. The real inversion plane. In a real Euclidean plane an inversion has been seen to be a one-to-one and reciprocal transformation except in that it transforms l_∞ to the center of inversion, and the center to l_∞. An inversion, therefore, is strictly one to one if we regard it as a transformation of the set of objects composed of the points of the real Euclidean plane together with l_∞ regarded as a single object.

DEFINITION. The set of points in a real Euclidean plane, together with the line at infinity regarded as a single object, is called a *real inversion plane*; l_∞ is called the *point at infinity* of the inversion plane. The set of points on a real circle, or on a real line l together with l_∞, is called a *circle* of the inversion plane. An *inversion* is either an inversion in the sense of § 71 with respect to a real or imaginary circle or an orthogonal line reflection. Circular transformations, etc. are defined as in § 92. The set of theorems about the inversion plane, which remain valid when the figures to which they refer are subjected to every transformation of the inversion group, is called the *real inversion geometry.*

Although the point at infinity receives special mention in this definition, from the point of view of the inversion geometry it is not to be distinguished from any other point of the inversion plane. For any point of the inversion plane can be carried to any other point of it by an inversion. In a set of assumptions for the inversion geometry as a separate science, there would be no mention of a point at infinity; just as there is no mention of a line or a plane at infinity in our assumptions for projective geometry.

The inversion geometry has a relation to the Euclidean geometry which is entirely analogous to the relation of the projective geometry to the Euclidean; namely, the set of transformations of the inversion group which leaves one point of the inversion plane invariant is a parabolic metric group in the Euclidean plane obtained by omitting this point from the inversion plane.

A large class of theorems about circles can be stated with the utmost simplicity in terms of the geometry of inversion. For example, the propositions that three noncollinear ordinary points determine a circle and that two ordinary points determine a line combine into the single proposition:

THEOREM 19. *In the inversion plane any three distinct points are on one and but one circle.*

The theorem that there is one and only one circle touching a given circle C^2 at a given point A, and passing through a given point B not on C^2, may be put in the following form, which also includes the proposition that through a given point not on a given line l there is one and but one line parallel to l.

THEOREM 20. *There is one and but one circle through a point A on a circle C^2 and a point B not on C^2, and having no point except A in common with C^2.*

The theory of pencils of circles makes no special mention of the radical axis (§ 71), for the radical axis (with l_∞) is merely one circle of the pencil and is indistinguishable from the other circles. In like manner the center of a circle is not to be distinguished from any other point; for the center is merely the inverse of l_∞, with respect to the circle, and the inversion group does not leave l_∞ invariant.

Thus the theory of pencils of circles in the inversion geometry involves no reference to the radical axis or to the line of centers. A pencil of circles may be defined as follows:

DEFINITION. A *pencil of circles* is either (a) the set of all circles through two distinct points, or (b) the set of all circles orthogonal to the circles of a pencil of Type (a), or (c) the set of all circles through a point of a given circle C^2 and meeting C^2 in no other point. A pencil of circles is said to be *hyperbolic, elliptic,* or *parabolic,* according as it is of Types (a), (b), or (c). Any point common to all circles of a pencil is called a *base point* of the pencil.

By comparison with the theorems in the preceding sections it is evident that the pencils of circles of these three types include all the pencils referred to in § 71 and also certain pencils of circles which are regarded as degenerate, from the Euclidean point of view. Thus, consider a pencil of lines through an ordinary point of a Euclidean plane. Each of these lines, with l_∞, constitutes a degenerate circle, and the set of degenerate circles is a pencil according to the definition above. Again, a pencil of parallel lines in the Euclidean plane determines a set of circles $[K^2]$ in the inversion plane which have in common only the one point l_∞. By Theorem 11, Cor. 3, any inversion Γ with a center O transforms $[K^2]$ into a set of circles $[K_1^2]$

Since there is one and only one circle of the set $[K_1^2]$ through every point of the Euclidean plane, $[K_1^2]$ must be a pencil of circles of Type (c).

The fundamental theorems about circular transformations may be stated as follows:

THEOREM 21. *A circular transformation is a one-to-one transformation of the inversion plane which carries circles into circles. There is a unique direct circular transformation carrying three distinct points A, B, C to three distinct points A', B', C' respectively. A circular transformation leaving three points invariant is either an inversion relative to the circle through these three points or the identity.*

The theorems on orthogonal circles in § 71, together with the corresponding propositions on circles, lines, and orthogonal line reflections, become:

THEOREM 22. *Two circles are orthogonal if and only if one of them passes through two points which are inverse with respect to the other.*

COROLLARY 1. *Two circles are orthogonal if and only if they belong respectively to two pencils of circles such that the limiting points of one pencil are the common points of the circles of the other pencil.*

COROLLARY 2. *If A_1 and A_2 are inverse with respect to a circle C^2, all circles through A_1 and orthogonal to C^2 pass through A_2.*

The correspondence Γ, which was established in §§ 90, 91, between the Euclidean plane and the complex projective line, is one to one and reciprocal between the inversion plane and the complex line. Since circles and chains correspond under Γ, the inversion geometry is identical with the geometry of chains on a complex line. The direct circular transformations of the inversion plane correspond to the projectivities of the complex line.

It follows from § 90 that the inversion with respect to the chain $\mathsf{C}(Q_0Q_1Q_\infty)$ transforms every point $z=x+iy$ into the conjugate imaginary point $\bar{z}=x-iy$. Hence an inversion with regard to any chain is a transformation projectively equivalent to that by which each point goes to its conjugate imaginary point (cf. § 78). For this reason we make the definition:

DEFINITION. Two points are said to be *conjugate* with respect to

It is easily seen that any nondirect circular transformation is a product of a particular inversion and a direct circular transformation Hence any nondirect transformation may be written in the form

$$z' = \frac{a\bar{z} + b}{c\bar{z} + d}.$$

We shall return to this subject in § 99.

EXERCISES

1. Construct a set of assumptions for the inversion geometry as a separate science.*

2. Work out the theorems analogous to those of §§ 71, 90–96 for the parabolic metric group in a modular space. Thus obtain a modular inversion geometry. The number of points in a finite inversion plane is $p^2 + 1$ if the number of points on a circle is $p + 1$.

3. The double points of an involution leaving a chain invariant are inverse

n plane. The more elemen-
version plane follow readily
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that the order relations of
C $(Q_0 Q_1 Q_\infty)$ and the circle
on circles are unaltered by

circular transformations, and order relations on chains are unaltered by projectivities, it follows that Γ is such that the order relations of corresponding sets of points on any chain and the corresponding circle are identical. Therefore the theory of order in the inversion plane applies also to the complex line.

Returning to the Euclidean plane π', we know by § 28 that the points not on an ordinary line l fall into two classes such that any two points of the same class are joined by a segment not meeting l, whereas a line joining two points of different classes always meets l. By § 64 any circle containing two points of different classes meets l in two points. We thus have

THEOREM 23. DEFINITION. *The points of an inversion plane not on a circle C^2 fall into two classes, called* the two sides of C^2, *such that two points on the same side of C^2 are joined by a segment of a circle which does not contain any point of C^2, and such that any circle containing two points on different sides of C^2 contains two points of C^2.*

Since order relations on circles are not altered by inversion, there follows:

COROLLARY 1. *If two points are on opposite sides of a circle C^2, the points to which they are transformed by an inversion Π are on opposite sides of $\Pi(C)$.*

On a complex line the points on one side of the chain $\mathsf{C}(Q_0Q_1Q_\infty)$ are evidently those whose coördinates relative to the scale Q_0, Q_1, Q_∞ are $x+iy$, where x is real and y real and positive, and those on the other side are those whose coördinates are $x-iy$. Hence, in general,

COROLLARY 2. *The points D and D' are on opposite sides of a circle through A, B, C if and only if y and y' are of opposite sign in the following two equations:*

$$R(AB, CD) = x+iy, \qquad R(AB, CD') = x'+iy',$$

where x, y, x', y' are all real.

DEFINITION. A throw $\mathsf{T}(AB, CD)$ is said to be *neutral* if $R(AB, CD)$ is real. Two throws $\mathsf{T}(AB, CD)$ and $\mathsf{T}(A'B', C'D')$ are *similarly* or *oppositely sensed* according as y and y' are of the same or of opposite signs in the equations

$$R(AB, CD) = x+iy \quad \text{and} \quad R(A'B', C'D') = x'+iy',$$

x, y, x', y' being real.

From this definition it is obvious that a direct circular transformation transforms any non-neutral throw into a similarly sensed throw. It is also obvious that an inversion which reduces in the Euclidean plane π to an orthogonal line reflection changes non-neutral throws into oppositely sensed throws. Hence we have

THEOREM 24. *A direct circular transformation carries non-neutral throws into similarly sensed throws, and a nondirect circular transformation carries them into oppositely sensed throws.*

EXERCISES

1. Two circles C^2, K^2 intersecting in two distinct points separate the inversion plane into four classes of points such that two points of the same class are joined by a segment of a circle containing no points of C^2 and K^2, whereas any circle containing points of different classes contains points of C^2 and K^2.

2. Two points which are inverse with respect to a circle are on opposite sides of it.

3. What is the relation between the sense of throws as defined above and the sense of noncollinear point triads in a Euclidean plane as defined in § 30?

4. In a Euclidean plane if a triangle ABC is carried to a triangle $A'B'C'$ by an inversion, the sense $S(ABC)$ is the same as or different from $S(A'B'C')$ according as the center of the inversion is or is not interior to the circle ABC.

5. In the notation of Ex. 7, § 92, if O is interior to a circle C^2, then O' is interior to $\Pi(C^2)$, and every point interior to C^2 is transformed by Π to a point exterior to O'.

98. Types of circular transformations. By § 5 every projectivity on a complex line has one or two double points. On account of the correspondence Γ the same result holds for the direct circular transformations of the real inversion plane.

Let us consider first a transformation Π having but one double point. In the theory of projectivities such a transformation has been called parabolic; and it has been proved that there is one and but one parabolic projectivity leaving a point M invariant and carrying a point A_0 to a point A_1. We have also seen that if A_{-1} is the point which goes to A_0, $\mathrm{R}(MA_0, A_1A_{-1}) = -1$. Hence A_{-1}, A_0, A_1 are on the same chain through M. Since A_{-1}, A_0, M are transformed into A_0, A_1, M respectively, this chain is left invariant by Π.

In like manner any other point B_0 not on the chain $\mathsf{C}(A_0A_1M)$ determines a chain which is left invariant by Π. These two chains cannot have another point than M in common, because this point

would have to be left invariant by Π. Thus Π leaves invariant a set of chains through M no two of which have a point in common, and such that there is one and only one chain of the set through any point except M.

If Π be regarded as a transformation of the inversion plane, this means that Π leaves invariant each circle of a pencil of circles of the parabolic type. In the Euclidean plane ϵ, obtained by leaving M out of the inversion plane, this pencil of circles is a system of parallel lines and Π is a direct similarity transformation. Now let us regard ϵ from the projective point of view. The transformation Π leaves all points of the line at infinity of ϵ invariant, because it leaves each of the circular points invariant as well as the point at infinity of the system of parallel lines. Hence Π is a translation in the Euclidean plane ϵ.

This result may be expressed in terms of the i
follows:

THEOREM 25. *Any direct*
invariant point transforms into itself every pencil of circles of the parabolic type having this point as base point. One and only one of these pencils is such that each circle of the pencil is invariant.

Returning to the Euclidean plane we have

THEOREM 26. *Any direct similarity transformation which is not a translation or the identity leaves invariant one and only one ordinary point.*

Proof. Regard the Euclidean plane as obtained by omitting one point from an inversion plane. A direct similarity transformation effects a transformation of the direct inversion group and leaves this point invariant. In case it leaves only this point invariant, it has just been seen to be a translation in the Euclidean plane. If not, by the first paragraph of this section it has one and only one other invariant point unless it reduces to the identity.

A similarity transformation leaving an ordinary point O invariant must transform into itself the pencil of lines through this point and the pencil of circles having this point as center.

Two important special cases arise, namely, a rotation about O and a dilation with O as center. Moreover, since there is one and only one direct similarity transformation leaving O invariant and carrying

as center invariant, and changes every line through O into another line through O. A dilation which is not a point reflection leaves every line through O invariant, and changes every circle with O as center into another such circle. Hence a product of a dilation and a rotation, neither of which is of period two, leaves invariant no line through O and no circle with O as center. Since either a rotation or a dilation of period two is a point reflection, any direct circular transformation falls under one of the three cases just mentioned or else is a point reflection. Stated in terms of the inversion plane these results become (cf. fig. 56, p. 158):

THEOREM 27. *A direct circular transformation having two fixed points transforms into itself the pencil of circles through the fixed points and also the pencils of circles about these points. The transformation either leaves invariant every circle of one pencil and no circle of the other pencil, or it leaves invariant no circle of either pencil, or it leaves invariant every circle of both pencils and is of period two.*

DEFINITION. A direct circular transformation is said to be *parabolic* if it leaves invariant only one point; to be *hyperbolic* if it leaves invariant two points and all circles through these points; to be *elliptic* if it leaves invariant two points and all circles about these points; to be *loxodromic* if it leaves invariant two points and no circle through the invariant points or about them.

The theorems above are all valid for the complex line if circles be replaced by chains and direct circular transformations by projectivities. The definition is to be understood to apply in the same fashion. Since every nonidentical projectivity on the complex line has one or two double points, the discussion above gives the theorem:

THEOREM 28. *A direct circular transformation* (*or a projectivity on a complex line*) *is either parabolic, hyperbolic, elliptic, or loxodromic.*

COROLLARY. *An involution on a complex line is both hyperbolic and elliptic; and any projectivity which is both hyperbolic and elliptic is an involution.*

EXERCISES

1. A projectivity whose double points x_1 and x_2 are distinct from each other and from the point P_∞ of a scale P_0, P_1, P_∞, and whose characteristic cross ratio (§ 73, Vol. I) is k, may be written

$$\frac{x' - x_1}{x' - x_2} = k\frac{x - x_1}{x - x_2}. \tag{10}$$

If one of the double points is P_∞ and the other is x_1, the projectivity may be written

$$x' - x_1 = k(x - x_1). \tag{11}$$

The projectivity is hyperbolic if k is real, elliptic if $k = e^{i\theta}$, where θ is real, and loxodromic if neither of these conditions is satisfied.

2. The parabolic projectivities with x_1 as double point may be written in the form

$$\frac{1}{x' - x_1} = \frac{1}{x - x_1} + at, \tag{12}$$

or, in case the double point is P_∞, in the form

$$x' = x + at.$$

In either case a subgroup is obtained by requiring t to be real. The locus of the points to which an arbitrary point is transformed by the transformation of this subgroup is a chain, and the set of such chains constitutes a parabolic pencil of chains.

3. The projectivities (10) and (11) for which

$$k = a^t,$$

where a is constant and t a real variable, form a group (a continuous group of one real parameter, in fact). The locus of the points to which a given point is carried by the transformations of this group or the group considered in Ex. 2 is called a *path curve*. In the nonparabolic cases, if a is real the path curves are chains through the double points. If a is complex and $|a| = 1$, they are chains about the double points. If a satisfies neither of these conditions, and the double points are P_0 and P_∞, the path curves are the loci of $x = re^{i\theta}$ satisfying the condition

$$r = \alpha e^{\beta\theta}, \tag{13}$$

where α and β are real constants; if the double points are not specialized, the path curves are projectively equivalent to the system (13). Diagrams illustrating the three types of path curves will be found in Klein and Fricke's Elliptische Modulfunktionen, Vol. I, Abschnitt II.

4. From the Euclidean point of view the r and θ in Ex. 3 are *polar coördinates*, and the loci (13) are *logarithmic spirals* meeting the lines through the origin at the angle $\tan^{-1}(1/\beta)$. (A generalization of the notion of angle

one-parameter group of Euclidean transformations may be a pencil of parallel lines or a pencil of concentric circles or a set of logarithmic spirals congruent to (13).

5. A projectivity having a finite period must be elliptic. A direct similarity transformation having a finite period must be a rotation.

6. A loxodromic projectivity is a product of an elliptic and a hyperbolic projectivity.

7. A projectivity leaving a chain invariant is either hyperbolic or elliptic.

99. Chains and antiprojectivities. The theory of chains on a complex line has been developed in the sections above by combining the general theory of one-dimensional projectivities with the Euclidean theory of circles. It is of course possible, and from some points of view desirable, to develop the theory of chains entirely independently of the Euclidean geometry. The reader is referred for the outlines of such a theory to an article by J. W. Young in the Annals of Mathematics, 2d Series, Vol. XI (1909), p. 33. Many of the properties of chains may be generalized to n dimensions, an *n-dimensional chain* or an *n-chain* being defined as a real n-dimensional space contained in an n-dimensional complex space in such a way that any three points on a line of the real space are on a line of the complex space. (This is the relation between S and S′ in §§ 6 and 70.) A discussion of the theory of these generalized chains will be found in the articles by C. Segre and C. Juel referred to below, and also in those by J. W. Young, Transactions of the American Mathematical Society, Vol. XI (1910), p. 280, and H. H. MacGregor, Annals of Mathematics, 2d Series, Vol. XIV (1912), p. 1.

The transformations,

$$z' = \frac{a\bar{z} + b}{c\bar{z} + d}, \qquad \begin{vmatrix} a & b \\ c & d \end{vmatrix} = 0, \tag{14}$$

of the complex line which were mentioned at the end of § 96 are analogous to the following class of transformations of the complex projective plane:

$$\begin{aligned} x_0' &= a_{00}\bar{x}_0 + a_{01}\bar{x}_1 + a_{02}\bar{x}_2, \\ x_1' &= a_{10}\bar{x}_0 + a_{11}\bar{x}_1 + a_{12}\bar{x}_2, \\ x_2' &= a_{20}\bar{x}_0 + a_{21}\bar{x}_1 + a_{22}\bar{x}_2, \end{aligned} \qquad \begin{vmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{vmatrix} \neq 0, \tag{15}$$

where $\bar{x}_i$ denotes the complex number conjugate to x_i. These trans-

to collinear points,* but they are not projective collineations. If x_0', x_1', x_2' be replaced by u_0', u_1', u_2', (15) gives the equation of a nonprojective correlation. The analogous formulas in four homogeneous variables will define nonprojective collineations and correlations in space.

DEFINITION. A nonprojective collineation or correlation or a one-dimensional transformation of the type (14) is called an *antiprojectivity.*

The theory of antiprojectivities has been studied by C. Juel, Acta Mathematica, Vol. XIV (1890), p. 1, and more fully by C. Segre, Torino Atti, Vol. XXV (1890), pp. 276, 430 and Vol. XXVI, pp. 35, 592. Their rôle in projective geometry may be regarded as defined by the following theorem due to G. Darboux, Mathematische Annalen, Vol. XVII (1880), p. 55. In this paper Darboux also points out the connection of the geometrical result with the functional equation,

$$f(x+y)=f(x)+f(y).$$

THEOREM 29. *Any one-to-one reciprocal transformation of a real projective line which carries harmonic sets into harmonic sets is projective.*†

Proof. Let Π be any transformation satisfying the hypotheses of the theorem, A, B, C any three points of the line, $\Pi(ABC)=A'B'C'$, and Π' the projectivity such that $\Pi'(A'B'C')=ABC$. Then $\Pi'\Pi(ABC)=ABC$. If we can prove that $\Pi'\Pi$ is the identity, it will follow that $\Pi=\Pi'^{-1}$, and hence that Π is a projectivity.

If $\Pi'\Pi$ were not the identity, it would transform a point P to a point Q distinct from P, while it left invariant all points of the net of rationality $\mathsf{R}(ABC)$. Let L_1, L_2, L_3 be points of this net in the order

$$\{PL_1L_2QL_3\}.$$

By Theorem 8, Chap. V, there would exist two real points S, T which harmonically separate the pairs PL_1 and L_2L_3. The transformation $\Pi'\Pi$ must carry S and T into two points harmonically separating the pairs QL_1 and L_2L_3. But since the latter two pairs separate each

* Cf. § 28, Vol. I.

† Von Staudt, Geometrie der Lage (Nürnberg, 1847), § 9, defined a projectivity of a real line as a transformation having this property. We are using Cremona's

other, by Theorem 8, Chap. V, there is no pair separating them both harmonically. Hence the assumption that $\Pi'\Pi$ is not the identity leads to a contradiction.

COROLLARY 1. *Any collineation or correlation in a real projective space is projective.*

Proof. Since a collineation transforms collinear points into collinear points, it transforms nets of rationality into nets of rationality in such a way that the correspondence between any two homologous

ive (cf. §§ 33–35, Vol. I). Hence, according to the theo-
correspondence effected by the collineation between
projective. Hence the collineation is projective.
nt proves that a correlation is projective. The reason-
hange in a real projective space of n dimensions.

reciprocal transformation of the
ts into points and circles into
rsion group.

Proof. Regard the inversion plane π, minus a point P_∞, as a Euclidean plane π'; let Π be any transformation satisfying the hypotheses of the corollary, let $\Pi(P_\infty) = P'$, and let Π' be an inversion carrying P' to P_∞. Then $\Pi'\Pi$ is a transformation satisfying the hypotheses of the corollary and leaving P_∞ invariant.

Since $\Pi'\Pi$ carries circles through P_∞ into circles, it effects a collineation in π. By the first corollary this collineation is projective. Since it carries circles into circles, it is a similarity transformation. Hence $\Pi'\Pi$ is a transformation, say Π'', of the inversion group in π'. Since $\Pi = \Pi'^{-1}\Pi''$, Π is also in the inversion group.

Translated into the geometry of the complex projective line the last corollary states:

COROLLARY 3. *Any transformation which carries chains into chains is either a projectivity or an antiprojectivity.*

In the light of Corollary 2 it is clear that the whole theory of the inversion group can be developed from the definition of a circular transformation as one which carries points into points and circles into circles. This is the point of view adopted by Möbius in his

EXERCISES

1. Derive the formulas for antiprojectivities in a modular geometry. Cf. O. Veblen, Transactions of the American Mathematical Society, Vol. VIII (1907), p. 366.

2. Which if any of the following propositions are true? Any one-to-one and reciprocal transformation of a complex projective line which carries harmonic sets of points into harmonic sets of points is either projective or antiprojective. Any one-to-one and reciprocal transformation of a complex projective line which carries quadrangular sets of points into quadrangular sets is either projective or antiprojective. Any collineation or correlation of a complex projective space is either projective or antiprojective.

3. An antiprojectivity carries four collinear points having an imaginary cross ratio into four points whose cross ratio is the conjugate imaginary.

100. Tetracyclic coördinates. The general equation of a circle in a Euclidean plane π with respect to the coördinate system employed in Chap. IV is

$$\alpha_0(x^2+y^2)+2\,\alpha_1 x+2\,\alpha_2 y+\alpha_3=0. \tag{16}$$

DEFINITION. A *degenerate circle* is either a pair of lines joining an ordinary point to the circular points at infinity or a pair of lines ll_∞, where l_∞ is the line at infinity.

Thus (16) represents a nondegenerate circle, provided that the following condition is not satisfied:

$$0=\begin{vmatrix}\alpha_3 & \alpha_1 & \alpha_2\\ \alpha_1 & \alpha_0 & 0\\ \alpha_2 & 0 & \alpha_0\end{vmatrix}\equiv\alpha_0(\alpha_0\alpha_3-\alpha_1^2-\alpha_2^2). \tag{17}$$

The condition $\alpha_0=0$ clearly means that (16) represents a degenerate circle consisting of l_∞ and an ordinary line, unless $\alpha_1=\alpha_2=0$ also, in which case (16) reduces to $\alpha_3=0$. The condition

$$\alpha_0\alpha_3-\alpha_1^2-\alpha_2^2=0 \tag{18}$$

means in case $\alpha_0\neq 0$ that (16) represents a pair of ordinary lines through the circular points. In case $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ are real, these two lines must be conjugate imaginaries. In the rest of this section the α's are supposed real.

Let us now interpret the ordered set of numbers $(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$ as homogeneous coördinates of a point in a projective space of three dimensions, S_3. For every point of S_3, except those satisfying (18), there is a unique circle or line pair ll_∞, where l is ordinary, and vice versa. Hence there is a one-to-one and reciprocal correspondence

between the points of S_3 not on the locus (18) and the circles of the inversion plane $\overline{\pi}$ obtained by adjoining l_∞ (regarded as a point) to π.

The points of S_3 which are on the locus (18) and not on $\alpha_0 = 0$ represent pairs of conjugate imaginary lines joining ordinary points of π to I_1 and I_2 respectively. There is one such pair of conjugate imaginary lines of π through each ordinary point of π. The points of S_3 on the locus (18) and not on $\alpha_0 = 0$ may therefore be regarded as corresponding to the points of $\overline{\pi}$, with the exception of l_∞. The only point of S_3 common to $\alpha_0 = 0$ and (18) is (0, 0, 0, 1), and this point may be taken to correspond to l_∞. Thus *the points of* S_3 *not on* (18) *represent circles of the inversion plane* $\overline{\pi}$, *and the points of* S_3 *on* (18) *represent the points of* $\overline{\pi}$.

Stated without the intervention of S_3, this means that the ordered set of numbers $(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$ taken homogeneously and subject to the relation (18) may be regarded as coördinates of the points of $\overline{\pi}$. When not subject to the relation (18) they may be regarded as coördinates of the circles and points in $\overline{\pi}$.

DEFINITION. The ordered sets of four numbers $(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$ subject to (18) are called *tetracyclic coördinates* of the points in $\overline{\pi}$. The same term is applied to any set of coördinates $(\beta_0, \beta_1, \beta_2, \beta_3)$ such that

$$\beta_i = \sum_{j=0}^{3} a_{ij}\alpha_j, \quad |a_{ij}| \neq 0. \qquad (i = 0, 1, 2, 3)$$

The circles (real or imaginary or degenerate) represented by (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1) are called the *base* or *fundamental* circles of the coördinate system.

A second particular choice of tetracyclic coördinates is given below.

The points of S_3 on (18) evidently constitute the set of all real points on the lines of intersection of corresponding planes of the two projective pencils

$$(19) \qquad \alpha_0 = \sigma(\alpha_1 + \sqrt{-1}\,\alpha_2) \quad \text{and} \quad \alpha_1 - \sqrt{-1}\,\alpha_2 = \sigma\alpha_3,$$

where the planes determined by the same value of σ are homologous. For (18) is obtained by eliminating σ between these two equations. The lines of intersection of homologous planes are all imaginary, but each contains one real point. This system of lines is, by § 103, Vol. I, a regulus, and the set of points on the lines, by § 104, Vol. I, a quadric surface. The locus (18) is therefore a real quadric surface all of whose rulers are imaginary (cf. also § 105, Vol. I).

sponds to a pencil of circles. For the points of the line joining $(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$ and $(\beta_0, \beta_1, \beta_2, \beta_3)$ correspond to the circles given by the equation

$$(\lambda\alpha_0+\mu\beta_0)(x^2+y^2)+(\lambda\alpha_1+\mu\beta_1)x+(\lambda\alpha_2+\mu\beta_2)y+(\lambda\alpha_3+\mu\beta_3)=0,$$

which represents a pencil of circles, together with its limiting points in case the latter are real.

Any collineation Γ of S_3 which carries the quadric (18) into itself must correspond to a transformation $\overline{\Gamma}$ of $\overline{\pi}$ which carries points into points, circles into circles, and pencils of circles into pencils of circles. $\overline{\Gamma}$ therefore has the property that if a point P of $\overline{\pi}$ is on a circle C^2 of $\overline{\pi}$, then $\overline{\Gamma}(P)$ is on $\overline{\Gamma}(C^2)$. By Theorem 29, Cor. 2, $\overline{\Gamma}$ is a circular transformation. Conversely, any circular transformation of $\overline{\pi}$ carries points to points, circles to circles, and pencils of circles to pencils of circles, and therefore corresponds to a collineation of S_3 which carries the quadric into itself. By Theorem 29, Cor. 1, this collineation is projective. In other words,

THEOREM 30. *The real inversion geometry is equivalent to the projective geometry of the quadric* (18).

COROLLARY. *The projective geometry of the real quadric* (18) *is equivalent to the complex projective geometry of a one-dimensional form.*

A one-to-one correspondence between a complex line and the real quadric (18) may also be set up as follows: Let l be any complex line in the regulus conjugate to that composed of the lines (19). Each of these lines contains one real point, P, of the quadric (18) and one point, Q, of l. The correspondence required is that in whic Q corresponds to P.

By properly choosing the constants which enter in the equation of a circle, we may set up the correspondence between the circles of the inversion plane and the points of an S_3 in such a way that the equation of the quadric surface corresponding to the points of the inversion plane has a particularly simple form. The equation of a circle in π may be written

$$(20) \quad \xi_0(x^2+y^2+1)+\xi_1(x^2+y^2-1)+2\,\xi_2 x+2\,\xi_3 y=0.$$

The points $(\xi_0, \xi_1, \xi_2, \xi_3)$ which correspond to points of the inversion plane now satisfy the equation

$$(21) \qquad \xi_0^2 = \xi_1^2 + \xi_2^2 + \xi_3^2,$$

and the circles corresponding to the four points (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), and (0, 0, 0, 1) are mutually orthogonal, one of them being imaginary. The coördinates $(\xi_0, \xi_1, \xi_2, \xi_3)$ are connected with $(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$ by the equations

$$\alpha_0 = \xi_0 + \xi_1, \quad \alpha_1 = \xi_2, \quad \alpha_2 = \xi_3, \quad \alpha_3 = \xi_0 - \xi_1,$$

which represent a collineation carrying the quadric (18) into the quadric (21).

If ξ_1/ξ_0, ξ_2/ξ_0, ξ_3/ξ_0 are regarded as nonhomogeneous coördinates with respect to a properly chosen frame of reference in a Euclidean space of three dimensions (cf. Chap. VII), (21) is the equation of a sphere. Hence the real inversion geometry is equivalent to the projective geometry of a sphere.

The latter equivalence may be established very neatly, with the aid of theorems of Euclidean three-dimensional geometry, by the method of stereographic projection. This discussion would naturally come as an exercise in the next chapter. It is to be found in books on function theory. On the whole subject of inversion geometry from this point of view, compare Bôcher, Reihenentwickelungen der Potentialtheorie (Leipzig, 1894), Chap. II.

DEFINITION. A circle C_3^2 is *linearly dependent* on two circles C_1^2 and C_2^2 if and only if it is in the pencil determined by C_1^2 and C_2^2. A circle C^2 is *linearly dependent* on n circles $C_1^2, \cdots, C_n^2$ if and only if it is a member of some finite set of circles $C_{n+1}^2, \cdots, C_{n+k}^2$ such that C_{n+i}^2 is linearly dependent on two of $C_1^2, \cdots, C_{n+i-1}^2$ $(i = 1, 2, \cdots, k)$. A set of n circles is *linearly independent* if no one of them is linearly dependent on the rest. The set of all circles linearly dependent on three linearly independent circles is called a *bundle.*

EXERCISES

1. The tetracyclic coördinates of a point are proportional to the powers of the point with respect to four fixed circles. If the four circles are mutually orthogonal, the identity which they satisfy reduces to (21).

2. A homogeneous equation of the first degree in tetracyclic coördinates represents a circle.

3. What kind of coördinates are obtained by taking as the base (a) two orthogonal circles and the two points in which they meet? (b) four points?

4. Two points of S_3 correspond to orthogonal circles if and only if they are

5. What set of circles corresponds to the conics in which the quadric (21) is met by the planes of a self-polar tetrahedron?

6. The direct circular transformations of $\bar{\pi}$ correspond to collineations of S_3 which leave each imaginary regulus of (21) invariant, while the others correspond to collineations interchanging the two reguli. The direct circular transformations of $\bar{\pi}$ correspond to direct collineations of S_3 in the sense of § 31, Chap. II.

7. The circles of a bundle correspond to the points of a plane of S_3.

8. The circles common to two bundles constitute a pencil and hence correspond to a line of S_3. Determine the projectively distinct types of pencils of circles on this basis.

9. All circles are linearly dependent on four linearly independent circles.

10. For any bundle of circles there is a point O which has the same power, c^2, with respect to every circle of the bundle. The radical axes of all pairs of circles in the bundle pass through O. In case there is more than one point O, the radical axes of all pairs of circles of the bundle coincide.

11. A bundle of circles may consist of all circles through a point (the set of all lines in a Euclidean plane is a special case of this). In every other case there is a nondegenerate circle orthogonal to all circles of the bundle. This circle has the point O (Ex. 10) as center and consists of the points C such that Dist $(OC) = c$. It is real if and only if c is real. In case c is imaginary let C^2 be the real circle consisting of points C' such that Dist $(OC') = c$; any circle of the bundle meets C^2 in the ends of a diameter.

101. Involutoric collineations. In view of the isomorphism between the real inversion group and the projective group of the real quadric (21), a further consideration of the group of a general quadric will be found apropos. In this connection we need to define certain particular types of involutoric collineations in any projective space. The theorems are all based on Assumptions A, E, P, H_0.

It is proved in § 29, Vol. I, that if ω is any plane and O any point not on ω, there exists a homology carrying any point P to a point P', provided that O, P, P' are distinct and collinear and P and P' are not on ω. It follows by the constructions given in that place that if one point P is transformed into its harmonic conjugate with regard to O and the point in which the line OP meets ω, every point is transformed in this way. It is also obvious that a homology is of period two if and only if it is of this type. Hence we make the following definition:

DEFINITION. A homology of a three-space is said to be *harmonic* if and only if it is of period two. A harmonic homology is also called a *point-plane reflection* and is denoted by $\{O\omega\}$ or $\{\omega O\}$, where O is the center and ω the plane of fixed points.

Definition. If l and l' are two nonintersecting lines of a projective space S_3, the transformation of S_3 leaving each point of l and l' invariant, and carrying any other point P to the point P' such that the line PP' meets l and l' in two points harmonically conjugate with regard to P and P', is called a *skew involution* or a *line reflection in l and l'*. It is denoted by $\{ll'\}$, and l and l' are called its *axes* or *directrices.*

Theorem 31. *A line reflection $\{ll'\}$ is a product of two point-plane reflections $\{O\omega\} \cdot \{P\pi\}$, where O and P are any two distinct points of l, ω is the plane on P and l', and π is the plane on O and l'.*

Proof. Consider any plane through l, and let L be the point in which it meets l'. In this plane $\{O\omega\}$ and $\{P\pi\}$ effect harmonic homologies whose centers are O and P respectively and whose axes are PL and OL respectively. The product is therefore the harmonic homology whose center is L and axis l. Hence the product $\{O\omega\} \cdot \{P\pi\}$ satisfies the definition of a line reflection whose axes are l and l'.

Corollary. *A line reflection is a projective collineation of period two, and any projective collineation of period two leaving invariant the points of two skew lines is a line reflection.*

EXERCISES

1. A projective collineation of period two in a plane is a harmonic homology.

2. A projective collineation of period two in a three-space is a point-plane reflection or a line reflection.

3. Let A, B, C, D be the vertices of a tetrahedron and α, β, γ, δ the respectively opposite faces. The transformations obtainable as products of the three harmonic homologies $\{A\alpha\}$, $\{B\beta\}$, $\{C\gamma\}$ constitute a commutative group of order 8 consisting of four point-plane reflections, three line reflections, and the identity. If the transformations other than the identity be denoted by 0, 1, 2, 3, 4, 5, 6, the multiplication table may be indicated by the modular plane given by the table (1) on p. 3, Vol. I, the rule being that the product of any two transformations corresponding to points i, j of the modular plane is the one which corresponds to the third point on the line joining i and j.

4. Generalize the last exercise to n dimensions. The group of involutoric transformations carrying $n+1$ independent points into themselves is commutative, and such that its multiplication table may be represented by means of a finite projective space of $n-1$ dimensions in which there are three points on each line.

5. A projectivity Γ of a complex line such that for one point P which is

characteristic cross ratio of Γ is an nth root of unity; in case $n = 3$, this cross ratio is said to be *equianharmonic*, and a set of four points having this cross ratio is said to be *equianharmonic*. As a transformation of the inversion group, Γ is equivalent to a rotation of period n.

6. A planar projective collineation of period n $(n > 2)$ is of Type I and the set of transforms of any point is on a conic, or else the collineation is a homology. In the first case, it is projectively equivalent to a rotation; in the second case, to a dilation (in general, imaginary). Consider the analogous problem in three dimensions. (For references on this and the last exercise cf. Encyclopédie des Sc. Math. III 8, § 14. The statements in the Encyclopédie on the planar case are not strictly correct, since they do not sufficiently take the existence of homologies of finite period into account.)

102. The projective group c[illegible] in § 104, Vol. I, a quadric may be regarded as the set of points of intersection of the lines of two conjugate reguli. These two reguli may be improper in the sense of Chap. IX, Vol. I, and in the following theorems improper elements are supposed adjoined when needed for the constructions employed.

Definition. If there are proper lines on a quadric, the quadric is said to be *ruled*, otherwise it is said to be *unruled.*

Theorem 32. *A harmonic homology whose center is the pole of its plane of fixed points with regard to a quadric surface Q^2 transforms Q^2 into itself in such a way that the two lines of Q^2 through any fixed point are interchanged.*

Proof. Let O be a point not on Q^2, and ω its polar plane. Any line l of Q^2 meets ω in a unique point K. The plane Ol contains one other line l' of Q^2, and (cf. § 104, Vol. I) l' passes through K. Any line joining O to a point L of l other than K must meet l' in a point L' such that L and L' are harmonically conjugate (§ 104, Vol. I) with regard to O and the point in which OL meets ω. Hence $\{O\omega\}$ interchanges l and l'. From this result the theorem follows at once.

Comparing Theorems 31 and 32, we have

Corollary. *A line reflection $\{ab\}$ such that a and b are polar with respect to a quadric Q^2 transforms Q^2 into itself in such a way that each regulus on Q^2 is transformed into itself.*

Theorem 33. *A projective collineation of a quadric which leaves three points of the quadric invariant, no two of the three points being on the same ruler, is either the identity or a harmonic homology whose center and plane of fixed points are polar with respect to the quadric.*

Proof. Denote the three points by A, B, C, the plane containing them by ω, and the pole of ω by O. Since no two of A, B, C are on a line of Q^2, ω contains no line of Q^2 and hence is not on O. Since three points of the conic in which ω meets the quadric are invariant, all such points are invariant, as is also O. Hence the given collineation is either the identity or a homology. In the latter case it must be a harmonic homology, since any two points of the quadric collinear with O are harmonically conjugate with respect to O and the point in which the line joining them meets ω.

THEOREM 34. *There exists one and only one projective collineation transforming each line of a regulus into itself and effecting a given projectivity on one of these lines. Such a collineation is a product of two line reflections whose axes are lines of the conjugate regulus.*

Proof. Let R_1^2 be a regulus and R_2^2 the conjugate regulus. A projectivity on a line, l, of R_1^2 is by § 78, Vol. I, a product of two involutions, say I and I$'$. Let $\{m_1m_2\}$ be a line reflection such that m_1 and m_2 are lines of R_2^2 through the double points of I, and let $\{m_1'm_2'\}$ be a line reflection such that m_1' and m_2' are lines of R_2^2 through the double points of I$'$. The product of $\{m_1'm_2'\}$ and $\{m_1m_2\}$ effects the given projectivity on l and transforms each line of R_1^2 into itself.

Conversely, any projectivity Γ leaving all lines of R_1^2 invariant effects a projectivity on l which is a product of two involutions I and I$'$. The line reflections $\{m_1m_2\}$ and $\{m_1'm_2'\}$ being defined as before,

$$\{m_1'm_2'\} \cdot \{m_1m_2\} \cdot \Gamma^{-1}$$

leaves all points of l invariant and hence leaves all lines of R_1^2 as well as all lines of R_2^2 invariant. Hence

$$\{m_1'm_2'\} \cdot \{m_1m_2\} \cdot \Gamma^{-1} = 1,$$

and

$$\{m_1'm_2'\} \cdot \{m_1m_2\} = \Gamma.$$

COROLLARY. *The group of permutations of the lines of a regulus effected by the projective collineations transforming the regulus into itself is simply isomorphic with the projective group of a line.*

DEFINITION. A collineation of a quadric which carries each regulus on the quadric into itself is said to be *direct.*

THEOREM 35. *There is one and but one direct collineation of a quadric surface Q^2 carrying an ordered triad of points of Q^2, no two of which are on a line of Q^2, to an ordered triad of points of Q^2 no two*

Proof. Let ABC and PQR be the given ordered triads of points, let a, b, c, p, q, r be the lines of one regulus through the points A, B, C, P, Q, R respectively, and let a', b', c', p', q', r' respectively be the lines of the conjugate regulus through the same points. By the last theorem there is a projective collineation Γ carrying a, b, c to p, q, r respectively while leaving all lines of the conjugate regulus invariant, and also a projective collineation Γ' carrying $a'b'c'$ to $p'q'r'$ respectively while leaving all of the lines a, b, c, p, q, r invariant. The product of Γ and Γ' carries A, B, C to P, Q, R respectively. That there is only one direct collineation having this effect is a corollary of Theorem 33.

Let R_1^2 be the regulus containing the lines a, b, c, and R_2^2 the regulus containing a', b', c'. The two collineations Γ and Γ' which have been used in the proof above are commutative as transformations of R_1^2 because Γ' leaves all lines of R_1^2 invariant, and are commutative as transformations of R_2^2 because Γ leaves all lines of R_2^2 invariant. Hence

$$\Gamma\Gamma' = \Gamma'\Gamma.$$

By Theorem 34, $\quad \Gamma\Gamma' = \{lm\} \cdot \{rs\} \cdot \{l'm'\} \cdot \{r's'\},$

where l, m, r, s are lines of R_1^2, and l', m', r', s' are lines of R_2^2. The collineations $\{rs\}$ and $\{l'm'\}$ are commutative for the same reason that Γ and Γ' are commutative. Hence

$$\Gamma\Gamma' = \{lm\} \cdot \{l'm'\} \cdot \{rs\} \cdot \{r's'\}.$$

The pairs lm and $l'm'$ are two pairs of opposite edges of a tetrahedron the other two edges of which may be denoted by a and b. The product $\{lm\} \cdot \{l'm'\}$ leaves each point of a and b invariant and is involutoric on each of the lines l, l', m, m'. Hence

$$\{lm\} \cdot \{l'm'\} = \{ab\}.$$

The lines a and b are polar with respect to R_1^2 because one of them is the line joining the point ll' to the point mm', and the other the line of intersection of the plane ll' with the plane mm' (cf. § 104, Vol. I).

In like manner $\quad \{pq\} \cdot \{p'q'\} = \{cd\},$

where c and d are polar with respect to R_2^2. Hence we have

THEOREM 36. *Any direct projective collineation* T *of a quadric surface is expressible in the form*

Since any line reflection whose axes are polar with respect to a quadric is a product of two harmonic homologies whose centers are polar to their planes of fixed points (cf. Theorem 31), the last theorem implies

Corollary 1. *Any direct projective collineation of a quadric is a product of four harmonic homologies whose centers are polar to their respective planes of fixed points.*

Corollary 2. *Any nondirect projective collineation of a quadric is a product of an odd number of harmonic homologies whose centers are polar to their respective planes of fixed points.*

Proof. If a projective collineation Γ interchanges the two reguli, and Λ is a harmonic homology of the sort described in the statement of the corollary, then $\Gamma\Lambda = \Delta$ is a projective collineation leaving each regulus invariant. By Cor. 1, Δ is a product of an even number of harmonic homologies of the required sort, and hence $\Gamma = \Delta\Lambda$ is a ... an odd number.

103. Real quadrics. The isomorphism between the real inversion group and the projective collineation group of the real quadric (or sphere) (21) may now be studied more in detail. Since a circular transformation leaving three given points of the inversion plane $\bar{\pi}$ invariant is the identity or an inversion (Theorem 21), and since a collineation of S_3 leaving three points of the quadric (21) invariant is the identity or a harmonic homology whose center is polar to its plane of fixed points, it follows that inversions in $\bar{\pi}$ correspond to homologies of S_3. Hence the direct circular transformations of $\bar{\pi}$ correspond to the direct collineations of S_3 transforming (21) into itself.

An involution in $\bar{\pi}$ is a product of two inversions whose invariant circles intersect and are perpendicular. To say that the invariant circles intersect and are perpendicular is to say that they intersect in such a way that one of the circles is transformed into itself by the inversion with respect to the other. Now suppose that $\{O\omega\}$ and $\{P\pi\}$ are the harmonic homologies corresponding to the two inversions. If the points of the quadric on the plane ω are to be transformed among themselves by $\{P\pi\}$, ω must pass through P. In like manner π must pass through O. Hence

$$\{O\omega\} \cdot \{P\pi\} = \{ll'\}$$

where l is the line OP, l' the line $\omega\pi$, and the lines l and l' are polar with respect to the quadric. Hence the involutions in the group of direct circular transformations correspond to the line reflections whose axes are polar with respect to (21).

Thus the theorem that any direct circular transformation of $\overline{\pi}$ is a product of two involutions is equivalent to Theorem 36 applied to the quadric (21). Since an involution in $\overline{\pi}$ always has two double points, we have the additional information, not contained in § 102, that every line reflection transforming the quadric (21) into itself has two and only two fixed points on the quadric. The line joining these two points is obviously one of the axes of the line reflection. Hence the line reflection has two real axes one of which meets the quadric (21) and the other of which does not.

These remarks are enough to show how the real inversion geometry can be made effective in obtaining the theory of the real quadric (21). We shall now show that any real nonruled quadric is projectively equivalent to the quadric (21), from which it follows that the real inversion geometry is equivalent to the projective geometry of any real nonruled quadric.

A nonruled quadric is obviously nondegenerate. In the complex space any two nondegenerate quadrics are projectively equivalent, because any two reguli are projectively equivalent. Since (18) represents a quadric, it therefore follows that every nondegenerate quadric may be represented by an equation of the second degree.

Now let Q^2 be any quadric whose polar system transforms real points into real planes, and let the frame of reference be chosen so that $(1, 0, 0, 0)$, $(0, 1, 0, 0)$, $(0, 0, 1, 0)$, and $(0, 0, 0, 1)$ are vertices of a real self-polar tetrahedron. The plane section by the plane $x_0 = 0$ must be a conic whose equation is of the form

$$a_1x_1^2 + a_2x_2^2 + a_3x_3^2 = 0,$$

and similar remarks can be made about the sections by the planes $x_1 = 0$, $x_2 = 0$, and $x_3 = 0$. From this it follows that Q^2 has the equation

$$a_0x_0^2 + a_1x_1^2 + a_2x_2^2 + a_3x_3^2 = 0, \tag{22}$$

where a_0, a_1, a_2, a_3 are real. The projective collineation

$$x_0' = \sqrt{|a_0|}x_0, \quad x_1' = \sqrt{|a_1|}x_1, \quad x_2' = \sqrt{|a_2|}x_2, \quad x_3 = \sqrt{|a_3|}x_3 \tag{23}$$

transforms Q^2 into a quadric having one of the following equations

$$x_0'^2 \pm x_1'^2 \pm x_2'^2 \pm x_3'^2 = 0.$$

Any one of the eight quadrics thus represented is obviously equivalent projectively to one of the following three:

$$x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0, \tag{24}$$

$$-x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0, \tag{25}$$

$$-x_0^2 - x_1^2 + x_2^2 + x_3^2 = 0. \tag{26}$$

It is also obvious that (24) is imaginary, that (26) has real rulers, and that (25) is equivalent to (21).

EXERCISES

1. Determine the types of collineations transforming into itself (1) a real unruled quadric, (2) a real ruled quadric, (3) an imaginary quadric having a real polar system.

2. Discuss the projective groups of the three types of quadrics enumerated in the last exercise.

104. The complex inversion plane. A projective plane may be obtained from a Euclidean plane (cf. Introduction, Vol. I) by adjoining ideal points and an ideal line in such a way as to make it possible to regard every collineation as a one-to-one reciprocal transformation of all points in the plane. In like manner the real inversion plane has been obtained from the real Euclidean plane by adjoining a single ideal point which serves as the correspondent of the center of each inversion. Similar considerations will now be adduced showing that an inversion in the complex plane may be rendered one to one and reciprocal by introducing *two intersecting ideal lines.*

In the complex projective plane an inversion has been seen (§ 94) to be a one-to-one reciprocal transformation of all points not on the sides of the singular triangle OI_1I_2, and to effect a projective transformation interchanging the pencil of lines on I_1 with the pencil of lines on I_2. In this projectivity the line I_1I_2 is homologous both with OI_1 and with OI_2.

In the Euclidean plane obtained by omitting the line I_1I_2 from the projective plane, it follows that the inversion is one to one and reciprocal except for points on the two minimal lines, p_0 and m_0, through O. Moreover, it effects a projective correspondence between the set of minimal lines $[p]$ parallel with and distinct from p_0 and the set of minimal lines $[m]$ parallel with and distinct from m_0.

The correspondence between any line p and the homologous line m is incomplete because there is no point on p corresponding to the intersection of m with p_0 and no point on m corresponding to the intersection of p with m_0. This correspondence, however, may be made completely one to one and reciprocal by introducing an ideal point M_∞ on m as the correspondent of the point pm_0 and an ideal point P_∞ on p as the correspondent of the point mp_0. In order to treat all the minimal lines symmetrically, ideal points P'_∞ and M'_∞ must be introduced on p_0 and m_0, respectively, as mutually corresponding points. Also one other ideal point O_∞ is introduced as the correspondent of O.

According to these conventions the line p_0 together with its ideal point P'_∞ is transformed into a set of points consisting of O_∞, M'_∞, and all the points M_∞. This set of points is therefore called an ideal line $\overline{m}_\infty$. In like manner the line m_0 together with its ideal point M_∞ is transformed into a set of points consisting of O_∞, P'_∞, and all the points P_∞; and this set of points is called an ideal line $\overline{p}_\infty$. The Euclidean plane with the lines $\overline{p}_\infty$ and $\overline{m}_\infty$ adjoined is called an inversion plane. Or to state the definition formally and without reference to a particular inversion:

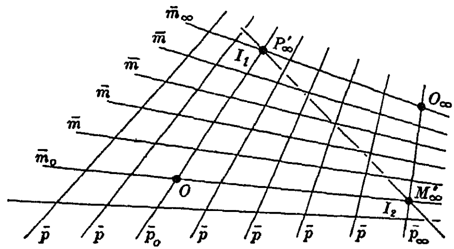

FIG. 74

DEFINITION. Given a complex Euclidean plane π and in it two pencils of minimal lines $[p]$ and $[m]$. By a *complex inversion plane* $\overline{\pi}$ is meant the set of all points of π (referred to as *ordinary points*) together with a set of elements called *ideal points* of which there is one, denoted by P_∞, for each p, and one, denoted by M_∞, for each m, distinct p's and m's determining distinct ideal points, and also one other ideal point which shall be denoted by O_∞. By a *minimal line* of $\overline{\pi}$ is meant (1) the set of points on a p together with the corresponding P_∞, or (2) the set of points on an m together with the corresponding M_∞, or (3) the set of all P_∞'s together with O_∞, or (4) the set of all M_∞'s together with O_∞. The minimal lines of Types (1) and (2) are called *ordinary*, and the lines (3) and (4) are called *ideal*.

A minimal line of Type (1) or (4) will be denoted by $\bar{p}$, of Type (2) or (3) by $\bar{m}$; the minimal lines of Types (3) and (4) are denoted by $\bar{m}_\infty$ and $\bar{p}_\infty$ respectively.

This definition is evidently such that each point of $\bar{\pi}$ is on a unique $\bar{p}$ and on a unique $\bar{m}$.

DEFINITION. By an *inversion* $\bar{\mathrm{I}}$ of $\bar{\pi}$ is meant a transformation defined as follows by an inversion I of π: If p_0 and m_0 are the singular lines of I, $\bar{\mathrm{I}}$ interchanges $\bar{p}_0$ with $\bar{m}_\infty$, $\bar{m}_0$ with $\bar{p}_\infty$, and each $\bar{p}$ containing a p with the $\bar{m}$ containing the m to which p is transformed by I. A point of $\bar{\pi}$ which is the intersection of a $\bar{p}$ and an $\bar{m}$ is transformed to the point which is the intersection of $\bar{\mathrm{I}}(\bar{p})$ and $\bar{\mathrm{I}}(\bar{m})$. The set of points of $\bar{\pi}$ left invariant by an inversion is called a *nondegenerate circle* of $\bar{\pi}$. A pair of minimal lines, one a $\bar{p}$ and the other an $\bar{m}$, is called a *degenerate circle* of $\bar{\pi}$.

By reference to § 94 it is evident that every circle of π is a subset of the points on a circle of $\bar{\pi}$.

The complex inversion plane is perhaps best understood by setting it in correspondence with a quadric surface, the lines of one regulus on the quadric being homologous with $[\bar{p}]$ and those of the other with $[\bar{m}]$. This correspondence may be studied by means of tetracyclic coördinates as in § 100, but it can also be set up by means of a geometric construction as follows:

Regard the complex Euclidean plane π with which we started as immersed in a complex Euclidean space. Let Q^2 be a quadric surface such that OI_1 is a line of one ruling and OI_2 of the other (fig. 74). Through I_1 and I_2 there are two other lines of the two rulings which intersect in a point O_∞. Any point P of the Euclidean plane is joined to O_∞ by a line which meets the quadric Q^2 in a unique point Q other than O_∞ and, conversely, any point of Q^2 which is not on either of the lines $O_\infty I_1$ or $O_\infty I_2$ is joined to O_∞ by a line which meets the Euclidean plane in a point P. Thus there is a correspondence T between the Euclidean plane and the points of Q^2 not on $O_\infty I_1$ or $O_\infty I_2$. This correspondence is such that every minimal line in π of the pencil on I_1 corresponds to a line of the quadric which is in the same ruling with OI_1, and every line of π of the pencil on I_2 corresponds to a line of the quadric which is in the same ruling with OI_2. From this it is evident that if ideal elements are adjoined to π as explained above, the ideal points can be regarded as corresponding to

the points of the lines $O_\infty I_1$ and $O_\infty I_2$ so that there is a one-to-one reciprocal correspondence between $\overline{\pi}$ and Q^2.

Now any nondegenerate circle of π is a conic through I_1 and I_2. This is projected from O_∞ by a cone of lines having in common with Q^2 the two lines $O_\infty I_1$ and $O_\infty I_2$. It follows that the cone and Q^2 have also a conic section in common. For let Q_1, Q_2, Q_3 be three of the common points which are not on the lines $O_\infty I_1$ and $O_\infty I_2$; the plane $Q_1 Q_2 Q_3$ meets the cone in a conic K_1^2 and Q^2 in a conic K_2^2. These two conics have also in common the points in which they meet the lines $O_\infty I_1$ and $O_\infty I_2$ (if these points coincide, K_1^2 and K_2^2 have a common tangent at this point), and hence $K_1^2 = K_2^2$. The conic K_1^2 is nondegenerate, because a nondegenerate cone through O_∞ can have no other line than $O_\infty I_1$ and $O_\infty I_2$ in common with Q^2. Hence every nondegenerate circle of π corresponds under T to a section of Q^2 by a nontangent plane.

Conversely, if K^2 is any nondegenerate conic section which is a plane section of Q^2, it is projected from O_∞ by a cone two of whose lines are $O_\infty I_1$ and $O_\infty I_2$. Hence K^2 corresponds under T to a nondegenerate circle of π.

An inversion in π with respect to a circle C^2 transforms every minimal line of the pencil $[p]$ into that one of $[m]$ which meets it on C^2. Let K^2 be the conic section on Q^2 corresponding under T to C^2. The inversion corresponds under T to a transformation of Q^2 by which every line of one regulus is transformed into the line of the other regulus which meets it in a point of K^2. This is the transformation (Theorem 32) effected by a harmonic homology whose plane of fixed points contains K^2 and whose center is the polar to this plane with respect to Q^2. Hence every inversion in π corresponds under T to a collineation of Q^2 effected by a harmonic homology whose center and plane of fixed points are polar with regard to Q^2. Conversely, every such collineation of Q^2 evidently corresponds under T to an inversion in π. Hence (Theorem 36, Cors. 1 and 2) the inversion group in π is isomorphic under T with the group of projective collineations of Q^2, and the direct circular transformations of π correspond to the projective collineations of Q^2 which carry each regulus into itself.

EXERCISE

Develop the theory of the modular inversion plane, using improper elements

the two variables x and y are thought of as completely independent of each other. The domain of each is the set of all complex numbers, including ∞. This domain is therefore equivalent to the complex line or to the real inversion plane. Thus the domain of x may be taken to be a real unruled quadric (in particular, a sphere) and the domain of y another real unruled quadric. Or the pair of values (x, y) may be regarded as an *ordered pair of points* on the same real unruled quadric.

Now consider a regulus in the complex projective space and, adopting the notation of the last section (fig. 74), let a scale be established on the lines $\overline{p}_0$ and $\overline{m}_0$ so that O is the zero in each scale. Let x be of any point on $\overline{p}_0$ and y of any point on $\overline{m}_0$. Then a unique point on the quadric, i.e. the $\overline{m}$ through the point with x as its coördinate, and the line $\overline{p}$ through the point with y as its coördinate. Conversely, the same construction determines a pair of numbers (x, y) for each point of the quadric.

Definition. The set of all ordered pairs (x, y) where x and y are complex numbers, including ∞, is called a *complex function plane,* or the *plane of the theory of functions* of complex variables, or the complex *plane of analysis.* The ordered pairs (x, y) are called *points.* Any point for which $x = \infty$ or $y = \infty$ is said to be *ideal* or *at infinity,* and all other points are called *ordinary.*

The points at infinity of the function plane can be represented conveniently by replacing x by a pair of homogeneous coördinates x_0, x_1 such that $x_1/x_0 = x$, and y by a pair (y_0, y_1) such that $y_1/y_0 = y$. Thus the points of the function plane are represented by

$$(x_0, x_1;\ y_0, y_1),$$

and the ideal points are those satisfying the condition

$$x_0 y_0 = 0.$$

The set of ordinary points of the function plane obviously forms a Euclidean plane in which a line is the locus of an equation of the form

$$ax + by + c = 0.$$

This is equivalent in homogeneous coördinates to

$$(27) \qquad a x_1 y_0 + b y_1 x_0 + c x_0 y_0 = 0,$$

an equation which is linear both in the pair of variables x_0, x_1 and in the pair y_0, y_1. The most general equation which is linear in both pairs is

$$(28) \qquad \alpha x_0 y_0 + \beta x_0 y_1 + \gamma x_1 y_0 + \delta x_1 y_1 = 0.$$

This reduces to (27) if the condition be imposed that the locus shall contain the point (∞, ∞) which in homogeneous coördinates is $(0, 1; 0, 1)$.

DEFINITION. The set of points of the function plane satisfying (28) is called a *circle* (or a *bilinear curve*), and any circle of the form (27) is called a *line*.

The group of transformations which is indicated as most important by problems of elementary function theory has the equations

$$(29) \qquad \begin{aligned} x' &= \frac{p_1 x + q_1}{r_1 x + s_1}, & \begin{vmatrix} p_1 & q_1 \\ r_1 & s_1 \end{vmatrix} &\neq 0, \\ y' &= \frac{p_2 y + q_2}{r_2 y + s_2}, & \begin{vmatrix} p_2 & q_2 \\ r_2 & s_2 \end{vmatrix} &\neq 0, \end{aligned}$$

or, in homogeneous coördinates,

$$(30) \qquad \begin{aligned} x_1' &= p_1 x_1 + q_1 x_0, & y_1' &= p_2 y_1 + q_2 y_0, \\ x_0' &= r_1 x_1 + s_1 x_0, & y_0' &= r_2 y_1 + s_2 y_0. \end{aligned}$$

This group of transformations clearly transforms circles into circles. The subgroup obtained by imposing the conditions,

$$r_1 = 0, \qquad r_2 = 0,$$

transforms lines into lines because it leaves (∞, ∞) invariant.

Returning to the interpretation of the coördinates x and y on a quadric, it is clear (cf. § 102) that every transformation (29) represents a direct collineation of the quadric, the formula in x determining the transformation of one regulus and the formula in y the transformation of the conjugate regulus. Hence the fundamental group of the function plane is isomorphic with the group of direct projective collineations of a quadric surface.

The parameters x and y which determine the points of a regulus may be connected with the three-dimensional coördinates $(\xi_0, \xi_1, \xi_2, \xi_3)$ by means of the following equations:

$$(31) \qquad \begin{aligned} \xi_0 &= x_1y_1 + x_0y_0, \\ \xi_1 &= x_1y_1 - x_0y_0, \\ \xi_2 &= x_1y_0 + x_0y_1, \\ \xi_3 &= i(x_1y_0 - x_0y_1), \end{aligned}$$

where $i^2 = -1$. For the set of all points $(\xi_0, \xi_1, \xi_2, \xi_3)$ given by these equations are the points on the quadric,

$$(21) \qquad \xi_0^2 = \xi_1^2 + \xi_2^2 + \xi_3^2.$$

Any plane section of this quadric is given by a linear equation in ξ_0, ξ_1, ξ_2, ξ_3, which by (31) reduces to a relation of the form (28) among the parameters x_0, x_1; y_0, y_1. Hence the circles of the function plane correspond to the plane sections of the quadric (21). In view of the relation already established between the groups it follows that the geometry of a quadric in a complex projective space is identical with that of a complex function plane. In view of § 104 both these geometries are identical with the complex inversion geometry.*

The complex projective plane may be contrasted with the complex inversion plane or function plane in an interesting manner as follows: The homogeneous coördinates $(\alpha_0, \alpha_1, \alpha_2)$ may be regarded as the coefficients of a quadratic equation

$$(32) \qquad \alpha_0z_0^2 + \alpha_1z_0z_1 + \alpha_2z_1^2 = 0.$$

Every such equation determines two and only two values of z_1/z_0, which may coincide or become infinite (if $\alpha_2 = 0$); and, moreover, two distinct points of the projective plane determine distinct quadratic equations and hence distinct pairs of values of z_1/z_0.

*If one were to confine attention to real values, the definition of the plane of analysis given above would determine a set of elements abstractly equivalent to a real ruled quadric. This is distinct from the real inversion plane, because the latter is equivalent to a real nonruled quadric. For the purposes of the theory of func-

The numbers (z_0, z_1) may be taken as homogeneous coördinates on a projective line. Thus there is a one-to-one and reciprocal correspondence between the points of a complex projective plane and the pairs of points on a complex projective line. It is important to notice that the pairs of points on the line *are not ordered pairs*, because a pair of values of z_1/z_0 taken in either order would be the pair of roots of the same quadratic.

Now representing the points of a complex line on a real unruled quadric (e.g. a sphere), we have that the projective plane is in one-to-one reciprocal correspondence with the *unordered* pairs of points of the quadric. On the other hand, we have already seen that the complex projective plane is in one-to-one reciprocal correspondence with the *ordered* pairs of points of the quadric. In either case the points of a pair may coincide.

For further discussion of the subject of this section see "The Infinite Regions of Various Geometries" by M. Bôcher, Bulletin of the American Mathematical Society, Vol. XX (1914), p. 185.

106. Projectivities of one-dimensional forms in general. The theorems of the last four sections have established and made use of the fact that the permutations effected among the lines of a regulus by projective collineations form a group isomorphic with the projective group of a line. Now a regulus is a one-dimensional form of the second degree,* and the notion of one-dimensional projective transformation has been extended to all the other one-dimensional forms (Chap. VIII, Vol. I, particularly § 76). It is therefore to be expected that an analogous extension can be made to the regulus. This we shall now make, but instead of dealing with the regulus in particular, we shall restate the old definition in a form which includes the cases where the regulus is in question.

DEFINITION. A correspondence between any two one-dimensional forms whose elements are of different kinds and not such that all elements of one form are on every element of the other form is said to be *perspective* if it is one-to-one and reciprocal and such that each element of either form is on the corresponding element of the other form.

*The one-dimensional forms of the first and second degrees in three-space are the pencil of points, the flat pencil of lines, the pencil of planes, the point conic, the line conic, the cone of lines, the cone of planes, and the regulus.

This covers the notion of perspectivity as defined in Vol. I between a pencil of points and a pencil of lines or between a pencil of lines and a point conic, etc. It also defines perspectivities between (1) the lines of a regulus and the points on a line of the conjugate regulus, (2) the lines of a regulus and the planes on a line of the conjugate regulus, (3) the lines of a regulus and the points of a conic which is a plane section of the regulus, (4) the lines of a regulus and the planes of a cone tangent to the regulus.

Definition. A correspondence between two one-dimensional forms or among the elements of a single one-dimensional form is *projective* if and only if it is the resultant of a sequence of perspectivities.

This definition comprehends that made in § 22, Vol. I, for forms of the first degree, and extended in § 76, Vol. I, so as to include those of

valent under duality to a point conic. In order

ion it is necessary to prove that it does not

lation of perspectivity between one-

In other words, we must prove

ne-dimensional forms of the first degree is projective according to the new definition only if it is projective according to the definition of § 22, Vol. I.

To prove this theorem it is sufficient to show that a sequence of perspectivities beginning and ending with forms of the first degree and involving forms of the second degree can be replaced by one involving only forms of the first degree. This follows directly from the fact that each one-dimensional form of the second degree is generated by projective one-dimensional forms of the first degree. For example, if a pencil of points $[P]$ is perspective with a regulus $[l]$ and the regulus with a point conic and the point conic with something else, it follows by the theorems of § 103, Vol. I, that $[P]$ is perspective with the pencil of planes $[ml]$, where m is a line of the conjugate regulus and $[ml]$ is perspective with the point conic. Thus the regulus $[l]$ in this sequence of perspectivities is replaced by the pencil of planes $[ml]$. In similar fashion it can be shown by a consideration of the finite number of possible cases that however a form of the second degree may intervene in a sequence of perspectivities, it can be replaced by a form or forms of the first degree. The enumeration of the possible cases is left to the reader, the argument required in each case being obvious.

spondences of any one-dimensional form with itself is isomorphic with the projective group of a line. For let Γ be any projectivity of a one-dimensional form F^2 of the second degree (e.g. a regulus), and let Π represent a perspectivity between F^2 and a one-dimensional form F^1 of the first degree (e.g. a line of the conjugate regulus). Then $\Pi\Gamma\Pi^{-1}$ is a projectivity of F^1. In like manner, if Γ' is a projectivity of F^1, $\Pi^{-1}\Gamma'\Pi$ is a projectivity of F^2. Hence Π establishes an isomorphism between the two groups.

***107. Projectivities of a quadric.** An involution on a regulus is the transformation of the lines of the regulus effected by a line reflection whose axes are the double lines of the involution. Since any projectivity of a regulus is a product of two involutions, it may be regarded as effected by a three-dimensional projective collineation which transforms the regulus into itself. Conversely, any direct projective collineation transforming a quadric into itself is a product of two line reflections (Theorem 36) each of which effects an involution on each of the reguli on the quadric.

This relation between the theory of one-dimensional projectivities and the projective group of a quadric may be used to obtain properties of the quadric analogous to the properties of conic sections studied in Chap. VIII, Vol. I. The discussion is based on Assumptions A, E, P, H_0, improper points being adjoined to the space whenever this is required for quadratic constructions.

In Chap. VIII, Vol. I, we have seen that any projectivity on a conic determines a unique point, the center of the projectivity, and that the axes of any two involutions into which the projectivity may be resolved pass through its center. If, now, a projectivity Γ be given on a regulus, any plane π meets the regulus in a conic C^2 on which is determined a projectivity Γ' having a point P as center. This determines a correspondence between the planes π and points P of space which is a null system (§ 108, Vol. I), and hence the axes of the involutions into which the projectivity Γ' can be resolved form a linear complex. The formal proof of this statement follows.

THEOREM 37. *For any nonidentical projectivity of a regulus there exists a linear complex of lines [l] having the property that if l_1 is any line of the complex not tangent to the regulus, there are three lines l_2,*

l_3, l_4 such that l_2 is polar to l_1 and l_3 to l_4 with respect to the regulus, and such that the collineation

$$\{l_1l_2\} \cdot \{l_3l_4\}$$

effects the given projectivity on the regulus. Moreover, every line l_1 having this property belongs to the complex, and so do l_2, l_3, l_4.

Proof. Let R^2 be a regulus and Γ a projectivity of R^2. If l_1l_2 and l_3l_4 are pairs of polar lines such that $\{l_1l_2\} \cdot \{l_3l_4\}$ effects the given projectivity on R^2, let π be any plane containing l_1 and not tangent to R^2. The projectivity Γ on R^2 is perspective with a projectivity Γ' on the conic C^2 in which π meets R^2. Moreover, $\{l_1l_2\}$ and $\{l_3l_4\}$ effect involutions on R^2 which are perspective with involutions I' and I'' on C'^2. Thus on C^2

$$\Gamma' = \mathrm{I}'\mathrm{I}''.$$

But (cf. § 77, Vol. I) l_1 is the axis of I' and hence passes through the center of Γ'. A similar argument shows that l_i $(i = 2, 3, 4)$ passes through the center of the projectivity perspective with Γ on the conic in which R^2 is met by any plane containing l_i and not tangent to R^2.

Hence all lines l_1, l_2, l_3, l_4 defined as above are contained in the set $[l]$ of all lines l such that if π is any plane on l and not tangent to R^2, l is also on a point P defined as follows: Let C^2 be the conic in which π meets R^2 and Γ' the projectivity on C^2 perspective with the projectivity Γ on R^2; then P is the center of Γ'.

The set $[l]$ obviously contains all lines tangent to R^2 at points of the double lines (if existent) of Γ. If l_1 is any other line of $[l]$ let π be a plane on l_1 and not tangent to R^2, let C^2 be the conic in which π meets R^2, and let Γ' be the projectivity on C^2 perspective with Γ. By § 79, Vol. I, and the definition of $[l]$, Γ' is a product of two involutions having l_1 and another line, l_3, as axes. Let l_2 and l_4 be the polars of l_1 and l_3 respectively. Then $\{l_1l_2\} \cdot \{l_3l_4\}$ effects the perspectivity Γ' on C and hence effects Γ on R^2. By the first paragraph of the proof l_2, l_3, l_4 are all lines of $[l]$. Hence all lines of $[l]$ have the property enunciated in the theorem. It remains to prove that $[l]$ is a linear complex.

By definition, if π is a plane not tangent to R^2 the lines of $[l]$ in π form a flat pencil. If π is tangent to R^2 let p be the line of R^2 on π and q the line of the conjugate regulus on π. In case p is a fixed line of Γ, the lines l on π are the tangents to R^2, i.e. the pencil of lines on π and the point pq. In case p is not a fixed line

a line of $[l]$. Any other line l_1 of $[l]$ in π must have a polar line l_2 passing through the point pq. Let Γ'' be the projectivity on q perspective with Γ. If Γ is effected by $\{l_1l_2\}\cdot\{l_3l_4\}$, then Γ'' is the product of two involutions, I' and I'', which are perspective with the involutions effected on R^2 by $\{l_1l_2\}$ and $\{l_3l_4\}$ respectively. Since l_2 must pass through the point pq, the latter is a double point of I'. But when Γ'' is expressed as a product of two involutions, one of these involutions is fully determined by one of its double points in case the latter is not a double point of Γ'' (cf. § 78, Vol. I). Hence the other double point, P, is fixed; and since l_1 must pass through it, it follows that all lines of $[l]$ on π pass through P. Moreover, it is evident that if l_1 is any line (except q) on π and P, l_2 its polar line, and $\{l_3l_4\}$ any line reflection effecting an involution on R^2 which is perspective with I'', the projectivity Γ is effected by $\{l_1l_2\}\cdot\{l_3l_4\}$. Hence $[l]$ contains all lines on π and P. Hence $[l]$ is a linear complex by Theorem 24, Chap. XI, Vol. I.

THEOREM 38. *A direct projectivity* Γ *of a quadric surface* Q^2 *which does not leave all lines of either regulus invariant determines a linear congruence of lines having the property that if* a_1 *is any line of the congruence not tangent to* Q^2 *there exist lines* a_2, b_1, b_2 *of the congruence such that*

$$(33)\qquad \Gamma=\{a_1a_2\}\cdot\{b_1b_2\}.$$

Moreover, each line a_1 *having this property belongs to the congruence, and so do* a_2, b_1, b_2.

Proof. Γ effects a projectivity on each regulus of Q^2, and each of these reguli by the last theorems determines a linear complex of lines. The two complexes are obviously not identical and hence have a linear congruence in common. Any line a_1 of this congruence is either tangent to Q^2, or such that there exist lines a_2, b_1, b_2 which are in both complexes and such that $\{a_1a_2\}\cdot\{b_1b_2\}$ effects the same projectivity as Γ on both reguli. Hence $\{a_1a_2\}\cdot\{b_1b_2\}=\Gamma$. Moreover, any a_1 for which a_2, b_1, b_2 exist satisfying this condition must, by the last theorem, belong to both complexes and hence belong to this congruence.

COROLLARY 1. *The congruence referred to in the theorem may be degenerate and consist of all lines on a point of* Q^2 *and on a plane tangent to* Q^2 *at this point; or it may be parabolic and have a line of the quadric as directrix; or it may be hyperbolic and have a pair*

Proof. Let C denote the congruence referred to in the theorem and let Π be the polarity by which every point is transformed into its polar plane with respect to Q^2. This polarity transforms any line a_1 of C into its polar line, and the latter, by the theorem, is in C. Hence Π transforms C into itself.

According to § 107, Vol. I, any congruence is either degenerate, parabolic, hyperbolic, or elliptic. If degenerate, it consists of all lines on a point R or a plane ρ, R being on ρ. If Π transforms such a congruence into itself, it must interchange R and ρ, and hence R must be on Q^2 and ρ tangent to Q^2 at R. The congruence C will be of this type if b_1 meets a_1 in a point of Q^2 and does not meet a_2.

If C is parabolic, its one directrix must be transformed into itself by Π, and hence must be a line of Q^2. This case arises if a_1, a_2, b_1, b_2 all meet the same line of Q^2 and do not meet any other line of Q^2.

If C is hyperbolic, Π must either leave the two directrices fixed individually or interchange them. In the first case each directrix must be a line of Q^2, which implies that a_1, a_2, b_1, b_2 all meet two lines of Q^2 and hence that all lines of one regulus are left invariant by Γ, contrary to hypothesis. Hence the second case is the only possible one. It occurs when a_1, a_2, b_1, b_2 do not all meet any line of Q^2, but are met by a pair of real lines.

If C is elliptic, it has two improper directrices * and the reasoning is the same as for the hyperbolic case.

Definition. A line l is said to *meet* or *to be met by* a pair of lines pq if and only if it meets both of them. A pair of lines lm is said to *meet* or *cross* a pair pq if both l and m meet pq.

EXERCISES

1. The lines which cross the distinct pairs of an involution on a regulus together with the lines tangent to the regulus at points of the double lines (if existent) of the involution form a nondegenerate linear complex.

2. If two pairs of polar lines, a_1a_2 and b_1b_2, of a regulus meet each other, the involutions effected by $\{a_1a_2\}$ and $\{b_1b_2\}$ are harmonic (commutative) and their double lines form a harmonic set.

* This may be proved as follows: Let l_1, l_2, l_3, l_4 be lines of C not on the same regulus. Any plane on l_4 meets the regulus R^2 containing l_1, l_2, l_3 in a conic, and l_4 meets this conic in two improper points P_1, P_2. The two lines of the regulus conjugate to R^2 which pass through P_1, P_2 meet l_1, l_2, l_3, l_4 and hence meet all lines of C.

3. Let Γ be a projectivity on a regulus R^2. A variable plane meets R^2 in a conic C^2 on which there is a projectivity Γ' perspective with Γ. The axes of the projectivities Γ' are lines of a linear congruence.

4. Enumerate the types of collineations leaving invariant a quadric (1) in the complex space, (2) in a real space, (3) in various modular spaces.

*108. Products of pairs of involutoric projectivities.

THEOREM 39. *A direct projective collineation of a quadric surface is a line reflection whose axes are polar, if it interchanges two points of the quadric which are not joined by a line of the quadric.*

Proof. Denote the collineation by Γ, the quadric by Q^2, the two reguli on it by R_1^2 and R_2^2, and the two points which Γ interchanges by A and B. Let a and b be the lines of R_1^2 on A and B respectively, and a' and b' those of R_2^2 on A and B respectively. Since Γ interchanges a and b it effects an involution on R_1^2, and since it interchanges a' and b' it effects an involution on R_2^2. Let l, m be the double lines of the involution on R_1^2, and p, q those of the involution on R_2^2. Γ is evidently the product of $\{lm\}$ by $\{pq\}$ and hence is a line reflection whose axes are the line joining the points lp and mq and the line joining the planes lp and mq. These two lines are polar with regard to Q^2.

THEOREM 40. *Two lines which are not on a quadric Q^2 and do not meet the same line of Q^2 are met by one and but one polar pair of lines.*

Proof. Let one of the given lines meet the quadric in A and A' and the other meet it in B and B'. By Theorem 35 there is a unique direct projective collineation of the quadric which carries A to A', A' to A, and B to B'. By Theorem 39 this is a line reflection $\{lm\}$ and l and m are polar with respect to Q^2. Since $\{lm\}$ transforms A to A', l and m both meet the line AA', and since $\{lm\}$ transforms B to B', l and m both meet the line BB'.

If there were another pair of polar lines l', m' meeting AA' and BB', $\{l'm'\}$ would interchange A and A' and B and B'. By Theorem 35 $\{lm\} = \{l'm'\}$.

COROLLARY. *Two lines which are not on a quadric Q^2 and do not meet the same line of Q^2 are met by two and only two lines which are conjugate to them both with regard to Q^2.*

Proof. This follows directly from the theorem, because two mutually polar lines a, b meeting two lines l and m are both conjugate to

THEOREM 41. *If a simple hexagon is inscribed in a quadric surface in such a way that no two of its vertices are on a line of the quadric, the three pairs of opposite edges are met each by a polar pair of lines, and these three polar pairs of lines are in the same linear congruence.*

Proof. Let $A_1B_2C_1A_2B_1C_2$ be the simple hexagon. By the last theorem the pair of opposite edges A_1B_2, A_2B_1 is met by a pair of lines c_1, c_2 which are polar with respect to the quadric. In like manner B_2C_1, B_1C_2 are met by a polar pair a_1, a_2, and C_1A_2, C_2A_1 ... Consider the product of line reflections,

$$\cdot \{a_1a_2\}.$$

C_2, $\{b_1b_2\}$ carries C_2 to A_1, and ... carries B_2 to C_1, $\{b_1b_2\}$ carries C_1 to A_2, and $\{c_1c_2\}$ carries A_2 to B_1. Hence Γ interchanges B_1 and B_2, and by Theorem 39 it is a line reflection. Denoting Γ by $\{d_1d_2\}$ we have

$$\{c_1c_2\} \cdot \{d_1d_2\} = \{b_1b_2\} \cdot \{a_1a_2\}.$$

By Theorem 38 the axes of the four line reflections in this equation are all lines of the same congruence.

In view of the corollaries of Theorems 38 and 40 this theorem may be restated in the following forms:

COROLLARY 1. *If a simple hexagon is inscribed in a quadric in such a way that no two of its vertices are on a line of the quadric, the three polar pairs of lines which meet the pairs of opposite edges are met by a polar pair of lines (which may coincide).*

COROLLARY 2. *If a simple hexagon is inscribed in a quadric surface in such a way that no two of its vertices are on a line of the quadric, each pair of opposite edges is met by a unique pair of lines conjugate to both edges, and the latter three pairs of lines are met by a pair of lines conjugate to each of them. The lines of the last pair may coincide.**

* Bulletin of the American Mathematical Society, Vol. XVI (1909), pp. 55 and 62. A theorem of non-Euclidean geometry from which this may be obtained by generalization has been given by F. Klein, Mathematische Annalen, Vol. XXII (1883), p. 248.

This theorem is closely analogous to Pascal's theorem on conic sections (Chap. V, Vol. I). In the Pascal hexagon the pairs of opposite sides determine three points A, B, C which are collinear. In the hexagon inscribed in a quadric they determine three pairs of lines a_1a_2, b_1b_2, c_1c_2 which are in a linear congruence. In case the vertices of the hexagon are coplanar, the theorem on the quadric reduces directly to Pascal's.

The Pascal theorem may be proved by precisely the method used above. For let $A_1B_2C_1A_2B_1C_2$ be a hexagon inscribed in a conic and let A be the point $(B_1C_2,\ C_1B_2)$, B be $(C_1A_2,\ A_1C_2)$, and C be $(A_1B_2,\ B_1A_2)$. Let $\{Aa\}$, $\{Bb\}$, and $\{Cc\}$ be the harmonic homologies effecting the involutions having A, B, C as centers. By construction the projectivity effected by $\{Cc\}\cdot\{Bb\}\cdot\{Aa\}$ on the conic carries B_1 to B_2, and B_2 to B_1, and hence is an involution. Denoting its center and axis by D and d, we have

$$\{Cc\}\cdot\{Bb\}\cdot\{Aa\} = \{Dd\}.$$

This implies
$$\{Bb\}\cdot\{Aa\} = \{Cc\}\cdot\{Dd\}.$$

By the theorems of Chap. VIII, Vol. I, the line AB is the axis of the projectivity effected by $\{Bb\}\cdot\{Aa\}$ and must contain C and D. Hence A, B, C are collinear.

Pascal's theorem is thus based on the proposition that the product of three involutions on a conic is itself an involution if and only if the centers of the three involutions are collinear, i.e. if and only if their axes are concurrent. Let us denote an involution whose double points are L and M by $\{LM\}$, as in Ex. 11, § 52. If the involution is represented on a conic, the double points are joined by the axis of the involution. The proposition above then takes the form: The product $\{L_3M_3\}\cdot\{L_2M_2\}\cdot\{L_1M_1\}$ is an involution if and only if the lines L_1M_1, L_2M_2, L_3M_3 concur. The concurrence of the three lines means either that the three point pairs have a point in common or that they are themselves pairs of an involution. Thus the theorem on involutions may be stated as follows:

THEOREM 42. *In any one-dimensional form a product of three involutions $\{L_1M_1\}$, $\{L_2M_2\}$, $\{L_3M_3\}$ is an involution in case the pairs of points L_1M_1, L_2M_2, L_3M_3 have a point in common or are pairs of an involution; and the product is not an involution in any other case.*

The double points of the involutions may be either proper or improper (real or imaginary). In order to state the result entirely in terms of proper elements, the involutions may be represented on a conic and the condition stated in terms of the concurrence of their axes, as above; or it may be expressed by saying that they all belong to the same pencil of involutions, or by saying that they are all harmonic to the same projectivity.

This theorem on involutions in a one-dimensional form is fundamental in the theory of those groups of projectivities, in a space of any number of

it is essentially the same as Theorem 8, Chap. IV, which was fundamental in the theory of the parabolic metric group in the plane. Corresponding theorems in the Euclidean geometry of three dimensions will be found in §§ 114 and 121, Chap. VII. The same principle appears as Theorem 27, Cor. 1, Chap. III, in connection with the equiaffine group.

These groups are all projective and on that account related to the projective group of a one-dimensional form. But the essential feature which they have in common is that *every transformation of each group is a product of two involutoric transformations of the same group.* On this account, even without their common projective basis, the geometries corresponding to these groups must have many features in common. In particular, whenever there is some class of figures such that if two of the figures are interchanged by a transformation, the transformation is of period two, there must exist a theorem analogous to Pascal's theorem. As examples of this may be cited Theorem 41 above; Ex. 6, § 80, Vol. I; Ex. 1, § 122, below; and in the list of exercises below, Ex. 4, referring to the group of point reflections and translations, Exs. 5, 6 referring to the Euclidean group in a plane, Ex. 7 referring to the equiaffine group. On this subject in particular and also on the general theory of groups generated by transformations of period two, the reader should consult a series of articles by H. Wiener in the Berichte der Gesellschaft der Wissenschaften zu Leipzig, Vol. XLII (1890), pp. 13, 71, 245; Vol. XLIII (1891), pp. 424, 644; and also the article by Wiener referred to in § 45, above. Cf. also § 80, Vol. I.

EXERCISES

1. (Converse of Theorem 41.) If the three pairs of opposite edges of a simple hexagon are met by three pairs of lines a_1a_2, b_1b_2, c_1c_2 in pairs of points which are harmonically conjugate to the pairs of vertices with which they are collinear, and if the lines a_1, a_2, b_1, b_2, c_1, c_2 are in the same linear congruence, then the vertices of the hexagon are on a quadric surface with regard to which a_1a_2, b_1b_2, c_1c_2 are polar pairs of lines.

2. Two pairs of lines which are polar with regard to the same regulus cannot consist of lines of a common regulus.

3. If two lines l and m are met by two pairs of lines which are polar with respect to a quadric, l and m are polar.

4. In a Euclidean plane let A, B, C be the three points of intersection of pairs of opposite sides of a simple hexagon. If A and B are mid-points of the sides containing them, and C is the mid-point of one side containing it, then C is also a mid-point of the other side containing it.

5. Let $A_1B_2C_1A_2B_1C_2$ be a simple hexagon in a Euclidean plane. If the perpendicular bisector of the point pair A_1B_2 coincides with that of A_2B_1, and the perpendicular bisector of B_2C_1 with that of B_1C_2, and the perpendicular bisector of C_2A_1 with that of C_1A_2, then the three perpendicular bisectors

joining their mid-points are concurrent.

109. Conjugate imaginary lines of the second kind. The theory of antiprojectivities (§ 99) and the extended theory of projectivities of one-dimensional forms (§ 106) will now enable us to complete the theory of conjugate imaginary elements in certain essential details which we were not ready to discuss in § 78. Let S′ be a complex projective space and let S be a three-chain of S′, i.e. a space related to S′ in the manner described in §§ 6 and 70, and let us use the definitions and notations of § 70. The simplest type of antiprojective collineation of S′ is given by the equations

$$x_0' = \bar{x}_0, \quad x_1' = \bar{x}_1, \quad x_2' = \bar{x}_2, \quad x_3' = \bar{x}_3. \tag{34}$$

The frame of reference is such that the points of S have real coördinates. The transformation changes each point

$$(\alpha_0 + i\beta_0,\ \alpha_1 + i\beta_1,\ \alpha_2 + i\beta_2,\ \alpha_3 + i\beta_3),$$

where the α's and β's are real, into the point

$$(\alpha_0 - i\beta_0,\ \alpha_1 - i\beta_1,\ \alpha_2 - i\beta_2,\ \alpha_3 - i\beta_3).$$

These two points if distinct are joined by the real line

$$(\alpha_0 + \lambda\beta_0,\ \alpha_1 + \lambda\beta_1,\ \alpha_2 + \lambda\beta_2,\ \alpha_3 + \lambda\beta_3)$$

and are the double points of the involution determined by the transformation of the parameter λ,

$$\lambda' = -\frac{1}{\lambda}.$$

Comparing with the definition of conjugate imaginary points in § 78, it is clear that (34) is the transformation by which every point of S′ goes to its conjugate imaginary point, the points of S being regarded as real.

From the fact that the transformation (34) leaves no imaginary point invariant, it follows that it cannot leave any imaginary line or plane invariant. For the real line through an imaginary point P of the given line or plane is left invariant by (34), and hence P would be left invariant by (34). On the other hand, (34) leaves every real

Hence (34) interchanges any element of S′ with the element which is its conjugate imaginary according to the definition of § 78.

The definition of § 78 defines the notion of conjugate imaginary elements for all one-dimensional forms of the first or second degrees, and the theorems of that section cover all cases except that of a pair of conjugate imaginary lines which are the double lines of an elliptic involution in the lines of a regulus.

DEFINITION. An imaginary line which is a double line of an elliptic involution in a flat pencil is said to be of *the first kind*, and one which is a double line of an elliptic involution in a regulus is said to be of *the second kind.*

THEOREM 43. *Any imaginary line is either of the first or of the second kind.*

Proof. Let l be an imaginary line. It cannot contain two real points, else it would be a real line (§ 70). Hence it contains one or no real point. In the first case let O be the real point on l, P one of the imaginary points on l, and $\bar{P}$ the imaginary point conjugate to P. The line $P\bar{P}$ is real, and hence the plane $OP\bar{P}$ is real. Hence by § 78 the lines OP and $O\bar{P}$ are the double lines of an elliptic involution in the pencil of real lines on the point O and the plane $OP\bar{P}$.

In the second case let P, Q and R be three points of l and let $\bar{P}$, $\bar{Q}$ and $\bar{R}$ be their respective conjugate imaginary points. The lines $P\bar{P}$, $Q\bar{Q}$, $R\bar{R}$ are real and no two of them can intersect, for if they did l would be on a real plane, and we should have the case considered in the last paragraph. Hence these lines determine a regulus R_1^2 in S. On the real line $P\bar{P}$ there is by § 78 an elliptic involution having P and $\bar{P}$ as its imaginary double points. Hence there is an elliptic involution in the regulus R_2^2, conjugate to R_1^2, having l as one double line and a line $\bar{l}$ through $\bar{P}$ as the other. The lines l and $\bar{l}$ are conjugate imaginary lines by definition, and satisfy the definition of imaginary lines of the second kind. Since (34) transforms each element into its conjugate element, it is clear that $\bar{l}$ contains $\bar{Q}$ and $\bar{R}$ as well as $\bar{P}$.

The system of real lines obtained by joining each point of l to its conjugate imaginary point on $\bar{l}$ is, by the reasoning above, a set of

lines of the real space S, no two of which intersect. Any four of them determine a linear congruence (§ 107, Vol. I) C in S and also a linear congruence $\bar{C}$ of S′. The congruence C has the property that each of its lines is contained in a line of $\bar{C}$, and $\bar{C}$ evidently is the set of all lines joining points of l to points of $\bar{l}$. Hence C is an elliptic congruence according to the definition of § 107, Vol. I, and consists of all real lines meeting l and $\bar{l}$. Hence the system of real lines joining points of l to their conjugate imaginary points is an elliptic congruence in S, or in other words:

THEOREM 44. *An imaginary line of the second kind is a directrix of an elliptic congruence.*

The observation, made in the argument above, that there is one line of a certain elliptic congruence through each point of an imaginary line of the second kind, shows that an elliptic congruence may be taken as a real image of a complex one-dimensional form. This of course implies that the whole of the real inversion geometry can be carried over into the theory of the elliptic congruence and *vice versa*. Cf. the exercises below.

The relations between the imaginary lines of the second kind and the regulus and elliptic congruence are fundamental in the von Staudt theory of imaginaries which has been referred to in § 6. In addition to the references given in that place, the reader may consult the Encyclopédie des Sciences Mathématiques, III 8, § 19, and III 3, §§ 14, 15.

EXERCISES

1. An elliptic congruence in a real space has a pair of conjugate imaginary lines of the second kind as directrices.

2. The correspondence by which each point of an imaginary line l corresponds to its conjugate imaginary point is an antiprojectivity between l and its conjugate imaginary line.

3. Under the projective group of a real space any imaginary point is transformable into any other imaginary point, any imaginary line of the first kind into any imaginary line of the first kind, and any imaginary line of the second kind into any imaginary line of the second kind; an imaginary line of the first kind is not transformable into one of the second kind.

4. There is a one-to-one reciprocal correspondence between the points of a complex line and the lines of an elliptic congruence in a real space in which the points of a chain correspond to the lines of a regulus. By means of this

5. Let S_3' be a three-dimensional complex space. Any five noncoplanar points of S_3' determine a unique three-chain, which is a real S_3. This S_3 is related to S_3' in the manner described in §§ 6 and 70. Through any point P of S_3' not on S_3, there is (§ 78) a unique line which contains a line of S_3 (i.e. a chain C_1) as a subset. On this chain C_1 there is a unique elliptic involution having P as a double point. Let $\bar{P}$ be the other double point of this involution. P and $\bar{P}$ are the conjugate imaginary points with regard to the real space S_3, and the transformation of S_3' by which each point P not on S_3 goes to $\bar{P}$, and each point on S_3 is left invariant, may be called a *reflection in the three-chain* S_3. Any transformation which is a product of an odd number of reflections in three-chains is an antiprojective collineation, and any transformation which is a product of an even number of reflections in three-chains is a projective collineation. Every collineation is expressible in this form.

110. The principle of transference. We have seen how the geometry of the inversion group in the plane, arising initially as an extension of the Euclidean group, is equivalent to the projective geometry of the complex line and also to that of a real quadric which may be specialized as a sphere. We have also seen the equivalence of the projective groups of all one-dimensional forms in any properly projective space. Since the regulus is a one-dimensional form, this gave a hold on the group of the general quadric. The latter group in a complex space has been seen to be isomorphic with the complex inversion group and also with the fundamental group of the function plane.

At each step we have helped ourselves forward by transferring the results of one geometry to another, combining these with easily obtained theorems of the second geometry, and thus extending our knowledge of both. This is one of the characteristic methods of modern geometry. It was perhaps first used with clear understanding by O. Hesse,* and was formulated as a definite geometrical principle (Uebertragungsprinzip) by F. Klein in the article referred to in § 34.

This principle of transference or of carrying over the results of one geometry to another may be stated as follows: *Given a set of elements* [*e*] *and a group G of permutations of these elements, and a set of theorems* [*T*] *which state relations left invariant by G. Let* [*e'*] *be another set of elements, and G' a group of permutations of* [*e'*]. *If there is a one-to-one reciprocal correspondence between* [*e*] *and* [*e'*]

in which G is simply isomorphic with G', the set of theorems $[T]$ determines by a mere change of terminology a set of theorems $[T']$ which state relations among elements e' which are left invariant by G'.

This principle becomes effective when the method by which $[e]$ and G are defined is such as to make it easy to derive theorems which are not so easily seen for $[e']$ and G'. This has been abundantly illustrated in the present chapter, but the series of geometries equivalent to the projective geometry on a line could be much extended. Some of the possible extensions are mentioned in the exercises below.

From the example of the conic and the quadric surface (§ 107) it is clear that in order to carry results over from the theory of a set $[e]$ and a group G to a set $[e']$ and a group G' it is not necessary that the correspondence be one-to-one. The transference of theorems is, however, no longer a mere translation from one language, as it were, to another, but involves a study of the nature of the correspondence.

DEFINITION. Given a set of elements $[e]$ and a group G of permutations of $[e]$, the set of theorems $[T]$ which state relations among the elements of $[e]$ which are left invariant by G and are not left invariant by any group of permutations containing G is called a *generalized geometry* or *a branch of mathematics.**

This is, of course, a generalization of the definition of a geometry employed in §§ 34 and 39. At the time when the rôle of groups in geometry was outlined by Klein, the only sets $[e]$ under consideration were continuous manifolds, i.e. complex spaces of n dimensions or loci defined by one or more analytic relations among the coördinates of points in such spaces. The older writers restrict the term "geometry" by means of this restriction on the set $[e]$. But in view of the existence of modular spaces and other sets of elements determining sets of theorems more nearly identical with ordinary geometry than some of those admitted by Klein's original definition, it seems desirable to state the definition in the form adopted above.

In case the set of theorems $[T]$ is arranged deductively, as explained in the introduction to Vol. I, it becomes a mathematical science. The problem of the foundation of such a science is that of determining, if possible, a finite set of assumptions from which $[T]$ may be deduced.

* The generalized conception of a geometry is discussed very clearly in the article by G. Fano in the Encyclopädie der Math. Wiss. III AB 4*b*. A number of special

EXERCISES

1. If a projective collineation interchanges the two reguli on a quadric, homologous lines of the two reguli meet in points of a plane.

***2.** Let R^2 be a regulus, ω a plane not tangent to R^2, and O the pole of ω (ω may conveniently be regarded as the plane at infinity of a Euclidean space). A projectivity Γ of R^2 may be effected by a collineation Γ' leaving all lines of the conjugate regulus invariant. This collineation multiplied by the harmonic homology $\{O\omega\}$ gives a collineation Γ'' interchanging the two reguli. By Ex. 1, Γ'' determines a unique plane. Let P be the point polar to Γ'' with regard to R^2. The correspondence thus determined between the projectivities Γ of R^2 and the points of space not on R^2 is one to one and reciprocal. It is such that projectivities which are harmonic (§ 80, Vol. I) correspond to conjugate points with respect to R^2, and all the involutions correspond to points of ω.

***3.** The construction of Ex. 2 sets up a correspondence between the projectivities of a one-dimensional form and the points of a three-dimensional space which are not on a certain quadric. The same correspondence may be obtained by letting a projectivity

$$x' = \frac{a_0 x + a_1}{a_2 x + a_3}$$

correspond to the point (a_0, a_1, a_2, a_3). The relations between the one-dimensional and three-dimensional projective geometries thus obtained have been studied by C. Stéphanos, Mathematische Annalen, Vol. XXII (1883), p. 299.

***4.** Develop the theory of the twisted cubic curve in space along the following lines: (1) Define it algebraically. (2) Give a geometric definition. (3) Prove that Definitions (1) and (2) are equivalent. (4) Derive the further theorems on the cubic as far as possible from the geometric definition. It will be found that the properties of this cubic can be obtained largely from those of conic sections and one-dimensional projectivities in view of an isomorphism of the groups in question. The theorems should be classified according to the principle laid down in § 83.

***5.** A *rational curve* in a space of k dimensions is a locus given parametrically as follows:

$$x_0 = R_0(t), \quad x_1 = R_1(t), \cdots, \quad x_n = R_n(t),$$

where $R_0(t), \cdots, R_n(t)$ are rational functions of t. In case $k = n$ and the locus is not contained in any space of less than n dimensions, the curve is a *normal curve*. Develop the theories of various rational curves along the lines outlined in Ex. 4. For reference cf. § 28 of the encyclopedia article by Fano referred to above and articles by several authors in recent volumes of the American Journal of Mathematics.

***6.** The linear dependence of conic sections may be defined by substituting "point conic" or "line conic," as the case may be, for "circle" in the definition given at the end of § 100. Develop the theory of linear families of conics of one, two, three, and four dimensions, using the principle of correspondence whenever possible and classifying theorems according to the principle laid down in § 83. Cf. Encyclopédie des Sc. Math., III 18.

CHAPTER VII

AFFINE AND EUCLIDEAN GEOMETRY OF THREE DIMENSIONS

111. Affine geometry. DEFINITION. Let π_∞ be an arbitrary but fixed plane of a projective space S. The set of points of S not on π_∞ is called a *Euclidean space* and π_∞ is called the *plane at infinity* of this space. The plane π_∞ and the points and lines on π_∞ are said to be *ideal* or *at infinity*; all other points, lines, and planes of S are said to be *ordinary*. When no other indication is given, a point, line, or plane is understood to be ordinary. Any projective collineation transforming a Euclidean space into itself is said to be *affine*; the group of all such collineations is called the *affine group of three dimensions*, and the corresponding geometry the *affine geometry of three dimensions.*

DEFINITION. Two ordinary lines which have an ideal point in common are said to be *parallel* to each other. Two ordinary planes which have an ideal line in common, or an ordinary line and an ordinary plane which have an ideal point in common, are said to be *parallel* to each other.

In particular, a line or plane is said to be parallel to itself or to any plane or line which it is on. For ordinary points, lines, and planes we have as an obvious consequence of the assumptions and definitions of Chap. I, Vol. I, the following theorem:

THEOREM 1. *Through a given point there is one and only one line parallel to a given line. Through a given point there is one and only one plane parallel to a given plane. If two lines, l and l', are not in the same plane there is one and only one plane through a given point parallel to l and l'. If l and l' are parallel, any plane through l is parallel to l'.*

Another obvious though important theorem is the following:

THEOREM 2. *The transformations effected in an ordinary plane π by the affine group in space constitute the affine group of the Euclidean plane consisting of the ordinary points of π.*

In consequence of this theorem we have the whole affine plane geometry as a part of the affine geometry of three dimensions, and we shall take all the definitions and theorems of Chap. III for granted without further comment.

This discussion is valid for any space satisfying Assumptions A, E. The affine geometry of an ordered space (A, E, S) has already been considered in § 31, and certain additional theorems are given in Exs. 5–7 below.

EXERCISES

1. The lines joining the mid-points of the pairs of vertices of a tetrahedron

2. Classify the quadric surfaces from the point of view of real affine geometry. Develop the theory of diametral lines and planes. The real projective classification of the nondegenerate quadrics has been given in § 103. The affine classification is given in the Encyclopédie des Sc. Math. III 22, § 19.

***3.** Classify the linear congruences from the point of view of the real affine geometry. Cf. § 107, Vol. I.

***4.** Classify the linear complexes from the point of view of real affine geometry. Cf. § 108, Vol. I.

5. With respect to the coördinate system used in § 31 the points of the line joining $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$ are

$$\left(\frac{a_1 + \lambda b_1}{1+\lambda}, \frac{a_2 + \lambda b_2}{1+\lambda}, \frac{a_3 + \lambda b_3}{1+\lambda}\right),$$

B corresponding to $\lambda = \infty$ and the point at infinity to $\lambda = -1$. The segment AB consists of the points for which $\lambda > 0$ and its two prolongations of those for which $\lambda < -1$ and $-1 < \lambda < 0$ respectively.

6. Two points D and D' are on the same side of the plane ABC if and only if

$$S(ABCD) = S(ABCD').$$

7. Using the notation of § 101 and dealing with an ordered Euclidean space, $\{O\omega\}$ is an affine collineation which alters sense if O or ω is at infinity and $\{ll'\}$ is an affine collineation which does not alter sense if l or l' is at infinity. In an ordered projective space $\{ll'\}$ is, and $\{O\omega\}$ is not, a direct collineation.

112. Vectors, equivalence of point triads, etc. Definition. An elation having π_∞ as its plane of fixed points is called a *translation.* If l is an ordinary line on the center of the translation, the translation is said to be *parallel* to l.

The properties of the group of translations follow in large part from the following evident theorem.

THEOREM 3. *The transformations effected in an ordinary plane* π *by the translations leaving* π *invariant constitute the group of translations of the Euclidean plane composed of the ordinary points of* π.

As corollaries of this we have statements about translations in space which are verbally identical with Theorems 3–7, Chap. III. Theorem 8, Chap. III, generalizes as follows:

COROLLARY. *If* OX, OY, *and* OZ *are three noncoplanar lines and* T *any translation, there exists a unique triad of translations* T_x, T_y, T_z *parallel to* OX, OY, OZ *respectively and such that*

$$\mathrm{T} = \mathrm{T}_x\mathrm{T}_y\mathrm{T}_z.$$

The theory of congruence under translations generalizes to space without change, and the contents of §§ 39 and 40 may be taken as applying to the affine geometry in three-space. In like manner the definition of a field of vectors and of addition of vectors is carried over to space if the words "Euclidean plane" be replaced by "Euclidean space." The theorems of § 42 then apply without change.

We arrive at this point on the basis of Assumptions A, E, H_0. Adding Assumption P we take over the theory of the ratio of collinear vectors from §§ 43, 44. Some of the theorems to which it may be applied without essential modifications of the methods used in the planar case are given in the exercises below.

The definition of equivalence of ordered point triads in § 48 is such that if a plane π be carried by an affine collineation to a plane π', any two equivalent point triads of π are carried to two equivalent point triads of π'. Moreover, the definition of measure of ordered point triads in § 49 is such that if two coplanar ordered point triads ABC, DEF are carried by an affine collineation to $A'B'C'$, $D'E'F'$ respectively,

$$\frac{m\,(ABC)}{m\,(DEF)} = \frac{m\,(A'B'C')}{m\,(D'E'F')}. \tag{1}$$

This result in view of Theorem 39, Chap. III, depends on the corre-

DEFINITION. Two ordered point triads ABC and $A'B'C'$ are *equivalent* if and only if ABC may be carried by a translation to an ordered triad $A''B''C''$ which is equivalent in the sense of § 48, Chap. III, to $A'B'C'$.

The fundamental propositions with regard to equivalence, as developed in § 48, remain valid under the extended definition. Thus if $ABC \backsimeq A_1B_1C_1$ and $A_1B_1C_1 \backsimeq A_2B_2C_2$, $ABC \backsimeq A_2B_2C_2$; if $ABC \backsimeq A_1B_1C_1$, $A_1B_1C_1 \backsimeq ABC$, etc.

This extension of the notion of equivalence carries with it a cor-
ı of measure, i.e. measure is now
proviso that the unit triad in any
ƅ triad in any parallel plane.
of equivalence of ordered point
ıot generalize directly to the
ɔnal space.* We shall there-
measures of an ordered set of
› develop the corresponding
synthetic theory (cf. Ex. 13 below).

DEFINITION. By the *measure* of an ordered tetrad of points A_1, A_2, A_3, A_4 *relative to an ordered tetrad OPQR as unit* is meant the number

$$
(2) \qquad \begin{vmatrix} 1 & a_{11} & a_{12} & a_{13} \\ 1 & a_{21} & a_{22} & a_{23} \\ 1 & a_{31} & a_{32} & a_{33} \\ 1 & a_{41} & a_{42} & a_{43} \end{vmatrix} = m\,(A_1A_2A_3A_4),
$$

where (a_{i1}, a_{i2}, a_{i3}) are nonhomogeneous coördinates of A_i $(i = 1, 2, 3, 4)$ in a coördinate system in which O, P, Q, R are (0, 0, 0), (1, 0, 0), (0, 1, 0), (0, 0, 1) respectively. Two ordered tetrads are said to be *equivalent* if and only if they have the same measure. In real affine geometry the number $\frac{1}{6}|m\,(A_1A_2A_3A_4)|$ is called the *volume* of the tetrahedron $A_1A_2A_3A_4$ relative to the unit tetrahedron $OPQR$ and is denoted by $v\,(A_1A_2A_3A_4)$.

The theory of the equivalence of point pairs, triads, tetrads, etc. is the most elementary part of vector analysis and the Grassmann *Ausdehnungslehre*. This subject in particular, and the affine geometry

* Cf. M. Dehn, Mathematische Annalen, Vol. LV (1902), p. 465.

of three dimensions in general, is worthy of a much more extensive treatment than it is receiving here. We have referred only to that part of the subject which is essential to the study of the Euclidean geometry of three dimensions.

In the following exercises the coördinate system is understood to be that which is described in the definition of measure of ordered tetrads above. The vectors OP, OQ, OR are taken as units of measure for the respectively parallel systems of vectors. The ordered point triads OPQ, OQR, ORP are taken as units of measure for the respectively parallel systems of ordered point triads.

Definition. By the *projection* of a set of points $[X]$ on the x-axis is meant the set of points in which this axis is met by the planes through the points X and parallel to the plane $x=0$; and the projection on the y- and z-axes have analogous meanings.

By the *projection* of a set of points $[X]$ on the plane $x=0$ is meant the set of points in which this plane is met by the lines on points X and parallel to the x-axis; and the projections on the planes $y=0$ and $z=0$ have analogous meanings.

EXERCISES

1. The measures of ordered tetrads of points are unaltered by transformations

$$(3)\qquad \begin{aligned} x' &= b_{11}x + b_{12}y + b_{13}z + b_{10}, \\ y' &= b_{21}x + b_{22}y + b_{23}z + b_{20}, \\ z' &= b_{31}x + b_{32}y + b_{33}z + b_{30}, \end{aligned}$$

subject to the condition $\Delta = 1$, where

$$(4)\qquad \Delta = \begin{vmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{vmatrix}.$$

This group is called the *equiaffine* group and also the *special linear* group. The group for which $\Delta^2 = 1$ leaves volumes invariant.

2. Ratios of measures of ordered tetrads of points are left invariant by the affine group.

3. In an ordered space two ordered sets of points $ABCD$ and $A'B'C'D'$ are in the same sense or not according as $m(ABCD)$ and $m(A'B'C'D')$ have the same sign or not.

4. The product of two line reflections $\{ll'\}$ and $\{mm'\}$ (cf. § 101) is a translation if l' and m' are at infinity and l and m are parallel.

6. The projections of a point pair P_1P_2 on the x-, y-, and z-axes respectively have the measures

$$\alpha = x_2 - x_1, \quad \beta = y_2 - y_1, \quad \gamma = z_2 - z_1,$$

and those of the ordered point triad OP_1P_2 on the planes $x = 0$, $y = 0$, $z = 0$ respectively have the measures

$$\lambda = \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix}, \quad \mu = \begin{vmatrix} z_1 & x_1 \\ z_2 & x_2 \end{vmatrix}, \quad \nu = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}.$$

These numbers satisfy the relation

$$\alpha\lambda + \beta\mu + \gamma\nu = 0.$$

Any two points $P_1'P_2'$ of the line P_1P_2 such that Vect $P_1P_2 =$ Vect $P_1'P_2'$ deter- [illegible] $\beta, \gamma, \lambda, \mu, \nu$. These numbers are proportional to [illegible], Vol. I) of the line P_1P_2.

[illegible] Ex. 6, $\lambda = m(OPP_1P_2)$, $\mu = m(OQP_1P_2)$, [illegible] are the numbers analogous to α, β, γ, [illegible] P_4,

[illegible] $\gamma\nu' + \lambda\alpha' + \mu\beta' + \nu\gamma'$.

[illegible] f an ordered point triad $P_1P_2P_3$ on [illegible] re

[illegible] $\begin{vmatrix} y_3 & z_3 & 1 \end{vmatrix}$ [illegible] $\begin{vmatrix} x_3 & z_3 & 1 \end{vmatrix}$ $u_3 = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$.

The homogeneous coördinates of the plane $P_1P_2P_3$ are (u_0, u_1, u_2, u_3), where

$$u_0 = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = m(OP_1P_2P_3).$$

9. If P_1, P_2, P_3, P_4 are four noncoplanar points and P_3', P_4' are two points collinear with P_3 and P_4, then Vect $(P_3'P_4') =$ Vect (P_3P_4) if and only if $m(P_1P_2P_3P_4) = m(P_1P_2P_3'P_4')$.

10. If P_1, P_2, P_3, P_4 are four noncoplanar points and the lines P_1P_2, $P_1'P_2'$, $P_1''P_2''$ have a point in common and

$$\text{Vect}\,(P_1P_2) = \text{Vect}\,(P_1'P_2') + \text{Vect}\,(P_1''P_2''),$$

then
$$m(P_1P_2P_3P_4) = m(P_1'P_2'P_3P_4) + m(P_1''P_2''P_3P_4).$$

***11.** Study barycentric coördinates and the barycentric calculus for three-dimensional space. Cf. § 51, § 27, and references to Möbius in § 49.

***12.** Study the measure of n-points in space, generalizing the exercises in § 49.

***13.** Define two ordered tetrads $ABCD$ and $A'B'C'D'$ as equivalent provided that (1) $A = A'$, $B = B'$, $C = C'$, and the line DD' is parallel to the plane ABC, or (2) if there are a finite number of ordered tetrads $t_1, \cdots, t_n$

and t_n to $A'B'C'D'$. Develop a theory of equivalence as nearly as possible analogous to that of § 48. Show that two tetrads are equivalent in this sense if and only if they are equivalent according to the definition in the text.

***14.** An elation whose center is at infinity and whose plane of fixed points is ordinary is called a *simple shear*. The set of all products of simple shears is the equiaffine group. Develop the theory of the equiaffine group on this basis. Is it possible to generalize § 52 to space?

15. If a plane meets the sides $A_0A_1, A_1A_2, \ldots, A_nA_0$ of a simple polygon $A_0A_1A_2 \cdots A_n$ in points $B_0, B_1, \ldots, B_n$, respectively,

$$\frac{A_0B_0}{A_1B_0} \cdot \frac{A_1B_1}{A_2B_1} \cdots \frac{A_nB_n}{A_0B_n} = 1.$$

16. If a quadric surface (§ 104, Vol. I) meets the lines $A_0A_1, A_1A_2, \ldots, A_nA_0$ respectively in the pairs of points $B_0C_0, B_1C_1, \ldots, B_nC_n$, respectively,

$$\frac{A_0B_0}{A_1B_0} \cdot \frac{A_0C_0}{A_1C_1} \cdot \frac{A_1B_1}{A_2B_1} \cdot \frac{A_1C_1}{A_2C_1} \cdots \frac{A_nB_n}{A_0B_n} \cdot \frac{A_nC_n}{A_0C_n} = 1.$$

***17.** Six points of a plane no three of which are collinear satisfy the following identity:

$$m\,(123)\,m\,(456) - m\,(124)\,m\,(563) + m\,(125)\,m\,(634) - m\,(126)\,m\,(345) \equiv 0.$$

The ratio of any two terms in this sum is a projective invariant. These propositions are given by W. K. Clifford in the Proceedings of the London Mathematical Society, Vol. II (1866), p. 3, as the foundation of the theory of two-dimensional projectivities. Develop the details of the theory outlined by Clifford. Cf. also Möbius, Der barycentrische Calcul, § 221.

113. The parabolic metric group. Orthogonal lines and planes.
DEFINITION. Let Σ_∞ be an arbitrary but fixed polar system in the plane at infinity π_∞. This polar system shall be called the *absolute* or *orthogonal polar system*. The conic whose points lie on their polar lines with respect to Σ_∞ is, if existent, called the *circle at infinity*. The group of all collineations leaving Σ_∞ invariant is called the *parabolic metric group* and its transformations are called *similarity transformations*. Two figures conjugate under this group are said to be *similar*.

DEFINITION. Two ordinary planes or two ordinary lines are *orthogonal* or *perpendicular* if and only if they meet π_∞ in conjugate lines or points of the absolute polar system Σ_∞. An ordinary line and plane are *orthogonal* or *perpendicular* if and only if they meet π_∞ in a point and line which are polar with regard to Σ_∞. A line perpendicular to itself, i.e. a line through a point of the circle at infinity is

called a *minimal* or *isotropic line.* A plane perpendicular to itself, i.e. a plane meeting π_∞ in a tangent to the circle at infinity, is called a *minimal* or *isotropic plane.*

As the analogue of Theorems 2 and 3 we have

THEOREM 4. *The similarity transformations which leave an ordinary nonminimal plane π invariant, effect in π the transformations of a parabolic metric group in the Euclidean plane consisting of the ordinary points of π.*

Generalizing Theorem 1, Chap. IV, we have

THEOREM 5. *At every point O of a Euclidean space the correspondence between the lines and their perpendicular planes is a polar system, the projection of Σ_∞. All the lines through O perpendicular to a given*

cular to the given line at O; and all the

to a given plane are on the line through

If existent, the isotropic lines through a

and the isotropic planes through O the

f lines.

nonminimal planes meet in a nonminimal line, and two perpendicular nonminimal lines are parallel to a nonminimal plane.

COROLLARY 2. *If a plane 1 is perpendicular to a plane 2, and 2 is parallel to a plane 3, then 1 is perpendicular to 3. If a plane 1 is perpendicular to a line 2, and 2 is parallel to a line or plane 3, then 1 is perpendicular to 3. If a line 1 is perpendicular to a plane 2, and 2 is parallel to a line or plane 3, then 1 is perpendicular to 3. If a line 1 is perpendicular to a line 2, and 2 is parallel to a line 3, then 1 is perpendicular to 3.*

THEOREM 6. *Two nonparallel lines not both parallel to the same minimal plane are met by one and only one line perpendicular to them both; this line is not minimal.*

Proof. Let A_∞ and B_∞ be the points in which the given lines meet π_∞. By hypothesis $A_\infty \neq B_\infty$, and the line $A_\infty B_\infty$ is not tangent to the circle at infinity. Let C_∞ be the pole of the line $A_\infty B_\infty$ with respect to Σ_∞. The required common intersecting perpendicular is the line through C_∞ meeting the two given lines; this line is obviously unique and not minimal.

of the pairs of vertices meet in a point O. The line perpendicular to any face of the tetrahedron at the center of the circle through the three vertices in this face passes through O.

114. Orthogonal plane reflections. DEFINITION. A homology of period two whose center, P, is a point at infinity polar in the absolute polar system to the line at infinity of its plane of fixed points, π, is called an *orthogonal reflection in a plane* or an *orthogonal plane reflection* or a *symmetry with respect to a plane*, and may be denoted by $\{\pi P\}$.* The plane of fixed points is called the *plane of symmetry* of any two figures which correspond in the homology.

Since the center and the line at infinity of the plane of fixed points of an orthogonal reflection in a plane are pole and polar with respect to Σ_∞, we have

THEOREM 7. *An orthogonal reflection in a plane is a transformation of the parabolic metric group.*

By a direct generalization of Theorems 3 and 4, Chap. IV, we obtain the following:

THEOREM 8. (1) *If π and ρ are two parallel nonminimal planes, the product $\{\rho R\} \cdot \{\pi P\}$ is a translation parallel to any line perpendicular to π and ρ.* (2) *If* T *is a translation parallel to a nonminimal line l, π any plane perpendicular to l, and ρ the plane perpendicular to l passing through the mid-point of the point pair in which π and $\mathrm{T}(\pi)$ meet l, then*

$$\mathrm{T} = \{\rho R\} \cdot \{\pi P\};$$

and if σ is the plane perpendicular to l passing through the mid-point of the pair in which π and $\mathrm{T}^{-1}(\pi)$ meet l,

$$\mathrm{T} = \{\pi P\} \cdot \{\sigma S\}.$$

(3) *A translation parallel to a minimal line l is a product of four orthogonal plane reflections.*

THEOREM 9. *A product $\Lambda_n \Lambda_{n-1} \cdots \Lambda_1$ of orthogonal plane reflections is expressible in the form $\Lambda'_n \Lambda'_{n-1} \cdots \Lambda'_1 \mathrm{T}$ or $\mathrm{T}' \Lambda'_n \Lambda'_{n-1} \cdots \Lambda'_1$, where $\Lambda'_1, \Lambda'_2, \ldots, \Lambda'_n$ are orthogonal plane reflections whose planes of*

* In the rest of this chapter this notation will be used in the sense here defined and not in the more general sense of § 101.

fixed points all contain an arbitrary point O, and T *and* T′ *are translations. In case O is left invariant by* $\Lambda_n\Lambda_{n-1}\cdots\Lambda_1$, T *and* T′ *reduce to the identity.*

Proof. Let Λ_i' $(i=1, 2, \cdots n)$ denote the orthogonal plane reflection whose plane of fixed points is the plane through O parallel to the plane of fixed points of Λ_i. Then by Theorem 8, $\Lambda_i\Lambda_i' = \mathrm{T}_i$, T_i being a translation. Hence $\Lambda_i = \mathrm{T}_i\Lambda_i'$ and

$$(5) \qquad \Lambda_n\Lambda_{n-1}\cdots\Lambda_1 = \mathrm{T}_n\Lambda_n'\mathrm{T}_{n-1}\Lambda_n'\cdots\mathrm{T}_1\Lambda_1'.$$

By the generalization to space of Theorem 11, Cor. 2, Chap. III, if Σ is any affine collineation and T a translation, $\mathrm{T}\Sigma = \Sigma\mathrm{T}'$, where T′ is a translation. By repeated application of this proposition, (5) reduces to

$$\Lambda\ \Lambda \qquad \Lambda = \Lambda'\Lambda' \quad {}_1\cdots\Lambda_1'\mathrm{T} = \mathrm{T}'\Lambda_n'\Lambda_{n-1}'\cdots\Lambda_1',$$

he product $\Lambda_n\Lambda_{n-1}\cdots\Lambda_1$, since reflections Λ_i', it is left invariant and T′ reduce to the identity.

three orthogonal plane reflections line l, ordinary or ideal, the product $\Lambda_3\Lambda_2\Lambda_1$ *is an orthogonal plane reflection whose plane of fixed points contains l.*

Proof. One of the chief results obtained in Chap. VIII, Vol. I, can be put in the following form:* If T_1, T_2, T_3 are harmonic homologies leaving a conic invariant and such that their centers are collinear, $\mathrm{T}_3\mathrm{T}_2\mathrm{T}_1$ is a harmonic homology leaving the conic invariant. For by Theorem 19 of that chapter, and its corollary, the product $\mathrm{T}_2\mathrm{T}_1$ is expressible in the form $\mathrm{T}_3\mathrm{T}$, where T is a harmonic homology whose center and axis are polar with respect to the conic, the axis being concurrent with those of T_1, T_2, and T_3; and from $\mathrm{T}_2\mathrm{T}_1 = \mathrm{T}_3\mathrm{T}$ follows $\mathrm{T}_3\mathrm{T}_2\mathrm{T}_1 = \mathrm{T}_3\mathrm{T}_3\mathrm{T} = \mathrm{T}$.

Now if Λ_1, Λ_2, Λ_3 are orthogonal plane reflections whose planes of fixed points meet in an ordinary line l their centers are collinear. Hence they effect in π_∞ three harmonic homologies whose centers are the poles of their axes with respect to the absolute polar system and whose centers are collinear. Hence $\Lambda_3\Lambda_2\Lambda_1$ effects a harmonic homology in the plane at infinity and its axis, m_∞, passes

* Cf. the fine print in § 108.

through the point at infinity of l. Since l and m_∞ are both lines of fixed points of $\Lambda_3\Lambda_2\Lambda_1$, all points of the plane π containing l and m_∞ are invariant. Hence $\Lambda_3\Lambda_2\Lambda_1$ effects a homology having the pole of m_∞ with respect to Σ_∞ as center. Since this homology is of period two in π_∞ it must be an orthogonal plane reflection.

In case the planes of fixed points of $\Lambda_1, \Lambda_2, \Lambda_3$ are parallel we have by Theorem 8 (1) that $\Lambda_2\Lambda_1$ is a translation parallel to a line perpendicular to these planes, i.e. parallel to a nonminimal line. Hence by Theorem 8 (2) there exists an orthogonal plane reflection, Λ_4, such that

$$\Lambda_2\Lambda_1 = \Lambda_3\Lambda_4$$

or

$$\Lambda_3\Lambda_2\Lambda_1 = \Lambda_4.$$

COROLLARY. *If $\{\lambda_1 L_1\}$ and $\{\lambda_2 L_2\}$ are two orthogonal plane reflections, and λ'_1 is any ordinary nonminimal plane in the same pencil with λ_1 and λ_2, there exists a plane λ'_2 and points L'_1 and L'_2 such that*

$$\{\lambda_2 L_2\}\cdot\{\lambda_1 L_1\} = \{\lambda'_2 L_2\}\cdot\{\lambda'_1 L'_1\}.$$

Proof. By the theorem, if L'_1 is the point at infinity of a line perpendicular to λ'_1, there exists an orthogonal plane reflection $\{\lambda'_2 L'_2\}$ such that

$$\{\lambda_2 L_2\}\cdot\{\lambda_1 L_1\}\cdot\{\lambda'_1 L'_1\} = \{\lambda'_2 L'_2\},$$

and hence

$$\{\lambda_2 L_2\}\cdot\{\lambda_1 L_1\} = \{\lambda'_2 L'_2\}\cdot\{\lambda'_1 L'_1\}.$$

115. Displacements and symmetries. Congruence. We may now generalize directly from § 57, Chap. IV:

DEFINITION. A product of an even number of orthogonal plane reflections is called a *displacement* or *rigid motion*. A product of an odd number of orthogonal plane reflections is called a *symmetry*.

THEOREM 11. *The set of all displacements and symmetries is a self-conjugate subgroup of the parabolic metric group and contains the set of all displacements as a self-conjugate subgroup.*

DEFINITION. Two figures such that one can be transformed into the other by a displacement are said to be *congruent*. Two figures such that one can be transformed into the other by a symmetry are said to be *symmetric*.

THEOREM 12. *If a figure F_1 is congruent to a figure F_2, and F_2 to a figure F_3, then F_1 is congruent to F_3. If F_1 is symmetric with F_2, and F_2 with F_3, then F_1 is congruent to F_3. If F_1 is congruent to F_2, and F_2 symmetric with F_3, then F_1 is symmetric with F_3.*

THEOREM 15. *Any displacement leaving an ordinary point O inva-riant is a product of two orthogonal plane reflections whose planes o fixed points contain O.*

Proof. Consider a product of four orthogonal plane reflection whose planes of fixed points pass through O.

$$\Gamma = \{\lambda_4 L_4\} \cdot \{\lambda_3 L_3\} \cdot \{\lambda_2 L_2\} \cdot \{\lambda_1 L_1\}.$$

Let l be the line of intersection of λ_1 and λ_2, m that of λ_3 and λ and let λ be a plane containing l and m, where in case $l = m$, λ i chosen so as not to be minimal. If λ is nonminimal, by the corollar of Theorem 10 there exist orthogonal plane reflections $\{\mu M\}$, $\{\nu N$ such that

$$\{\lambda_2 L_2\} \cdot \{\lambda_1 L_1\} = \{\lambda L\} \cdot \{\mu M\},$$

and

$$\{\lambda_4 L_4\} \cdot \{\lambda_3 L_3\} = \{\nu N\} \cdot \{\lambda L\}.$$

Hence

$$\Gamma = \{\nu N\} \cdot \{\lambda L\} \cdot \{\lambda L\} \cdot \{\mu M\} = \{\nu N\} \cdot \{\mu M\}.$$

nal* $\{\lambda_1 L_1\}$ transforms λ to the other minima plane through l (i.e. the other plane containing l and a tangent to th circle at infinity), and $\{\lambda_2 L_2\}$ transforms this plane back to λ. In lik manner the product $\{\lambda_4 L_4\} \cdot \{\lambda_3 L_3\}$ leaves λ invariant. Hence λ i left invariant by Γ. On the other hand the line l is obviously no left invariant by Γ, and therefore Γ does not leave all points at infin ity invariant. Hence Γ leaves at most two tangents to the circle a infinity invariant, and thus leaves at most two minimal planes throug O invariant. Let λ_2' be any plane of the bundle containing λ_2 and λ which does not meet λ_1 in a line of an invariant minimal plane of Γ By the corollary of Theorem 10 there exists a plane λ_3' and points L and L_3' such that

$$\{\lambda_3 L_3\} \cdot \{\lambda_2 L_2\} = \{\lambda_3' L_3'\} \cdot \{\lambda_2' L_2'\},$$

and hence such that

$$\Gamma = \{\lambda_4 L_4\} \cdot \{\lambda_3' L_3'\} \cdot \{\lambda_2' L_2'\} \cdot \{\lambda_1 L_1\}.$$

Now let l be the line of intersection of λ_1 and λ_2', m that of λ_3' and λ and λ' the plane containing l and m. If λ' were minimal it would, a argued above for λ, be invariant under Γ, whereas λ_2' was so chose that l cannot be in such a plane. Hence the argument in the pre vious paragraph can be applied to the last expression obtained for Γ

* This case obviously does not arise in the real Euclidean geometry (§ 116), that this paragraph may be omitted if one is interested only in that case. It needed, however, in complex geometry.

Thus, in any case, a product of four orthogonal plane reflections whose planes of fixed points pass through O reduces to a product of two such reflections. By Theorem 9 any displacement leaving O invariant is a product of an even number, say $2n$, of orthogonal reflections in planes through O. This may be reduced to a product of two orthogonal reflections in planes through O by $n-1$ applications of the result proved above.

COROLLARY. *An orthogonal plane reflection is not a displacement.*

Proof. Let O be a point of the plane of fixed points of an orthogonal plane reflection Λ. If Λ were a displacement it would, by the theorem, be a product of two orthogonal plane reflections containing O and hence could only have a single line of fixed points.

DEFINITION. A displacement which is a product of two orthogonal plane reflections whose planes of fixed points have an ordinary line l in common is called a *rotation about* l, and l is called the *axis* of the rotation. If the axis is a minimal line the rotation is said to be *isotropic* or *minimal.*

THEOREM 14. *The product of two orthogonal reflections in perpendicular planes is a rotation of period two. It transforms every point P not on its axis to a point P' such that the axis is perpendicular to the line PP' at the mid-point of the pair PP'. It leaves invariant the points of its axis and the points in which any plane perpendicular to its axis meets the plane at infinity. Its axis cannot be a minimal line.*

Proof. Consider any plane π perpendicular to the planes of fixed points of the two orthogonal plane reflections Λ_1 and Λ_2. By the first corollary of Theorem 5 the axis of $\Lambda_2\Lambda_1$ is nonminimal and hence π is nonminimal. In π the transformations effected by Λ_1 and Λ_2 are orthogonal line reflections in the sense of Chap. IV, and their product is a point reflection (Theorem 5, Chap. IV) in the plane. From this the theorem follows in an obvious way.

DEFINITION. The product of two orthogonal reflections in perpendicular planes is called an *involutoric rotation* or an *orthogonal line reflection* or a *half turn.* If l is its axis and l' the polar with respect

THEOREM 15. DEFINITION. *The product of the orthogonal plane reflections in three perpendicular planes is a transformation carrying each point P to a point P' such that the point O of intersection of the three planes is the mid-point of the pair PP'. A transformation of this sort is called a* point reflection *or* symmetry *with respect to the point O as center. It is not a displacement. The points P and P' are said to be* symmetric *with respect to O.*

Proof. In the plane at infinity the three orthogonal plane reflections effect the three harmonic homologies whose centers and axes

ively opposite sides of a triangle. The

oints at infinity invariant. It also leaves

two on the line of intersection

of the orthogonal plane reflec-

tions. Hence it is a homology of period two with O as center and π_∞ as plane of fixed points. It is not a displacement, since by Theorem 13 a displacement leaving O invariant would have a line of fixed points passing through O.

THEOREM 16. *The transformations effected in a nonminimal plane π by the displacements leaving π invariant constitute the group of displacements and symmetries of the parabolic metric group whose absolute involution is that determined by Σ_∞ on the line at infinity of π.*

Proof. Let Γ be any displacement leaving π invariant, O an arbitrary point of π, and T the translation carrying O to $\Gamma(O)$. Then $\mathrm{T}^{-1}\Gamma(O) = O$, and hence, by Theorem 13, $\mathrm{T}^{-1}\Gamma$ is a rotation. Moreover, $\mathrm{T}^{-1}\Gamma$ leaves π invariant.

It is obvious from the definition of a rotation that it can leave π invariant only in case its axis is perpendicular to π or in case it is of period two and its axis is a line of π. If $\mathrm{T}^{-1}\Gamma$ falls under the first of these cases, it effects a rotation in π according to the definition of rotation in Chap. IV, and thus Γ effects a displacement in π. If $\mathrm{T}^{-1}\Gamma$ falls under the second of these cases it effects, and therefore Γ also effects, a symmetry in π according to the definition in Chap. IV.

COROLLARY 1. *The transformations effected in a nonminimal plane*

expressible in the forms

$$\Gamma = \mathrm{TP} \quad \text{and} \quad \Gamma = \mathrm{P'T'},$$

where T, T′ *are translations and* P, P′ *rotations leaving* O *invariant.*

Proof. As in the proof of the theorem above, let T be the translation carrying O to $\Gamma(O)$. Then $\mathrm{T}^{-1}\Gamma(O) = O$ and hence, by Theorem 13, $\mathrm{T}^{-1}\Gamma$ is a rotation, P. Hence $\Gamma = \mathrm{TP}$. If T′ is the translation carrying O to $\Gamma^{-1}(O)$, it follows in like manner that $\Gamma\mathrm{T'}(O)$ is a rotation P′ and hence that $\Gamma = \mathrm{P'T'}^{-1}$.

COROLLARY 3. *The transformations effected on a nonminimal line* p *by the displacements leaving* p *invariant constitute the group composed of all parabolic transformations and involutions leaving the point at infinity of* p *invariant.*

EXERCISES

1. Two point pairs are congruent if they are symmetric.

2. The set of all point reflections and translations forms a group which, unlike the analogous group in the plane (§ 45), is not a subgroup of the group of displacements. The product of two point reflections is a translation, and any translation is expressible as a product of two point reflections, one of which is arbitrary.

3. Study the theory of congruence in a minimal plane.

4. A rotation leaves no point invariant which is not on its axis. It leaves invariant all planes perpendicular to its axis and no others unless it is of period two, when it is an orthogonal line reflection.

116. Euclidean geometry of three dimensions. The last theorem may be regarded as the fundamental theorem of the parabolic metric geometry in space, for by means of it all the results of the two-dimensional parabolic metric geometry become immediately applicable.

Suppose now that we consider a three-space satisfying Assumptions A, E, H, C, R (or A, E, K), i.e. a real projective space. Suppose also that Σ_∞ be taken to be an elliptic polar system,* i.e. the polar system of an imaginary ellipse (§ 79). Then in any plane the parabolic metric geometry reduces to the Euclidean geometry and the displacements which leave this plane invariant are Euclidean displacements.

* The existence and properties of an elliptic polar system may be determined without recourse to imaginaries (in fact, on the basis A, E, P, S), as in § 89.

A set of assumptions for the Euclidean geometry of three dimensions is composed of I–XVI, given in §§ 29 and 66. We have seen in § 29 that I–IX are satisfied by a Euclidean space of three dimensions. Assumption XI is a consequence of Theorem 12, and Assumptions XII–XVI of Theorems 11 and 16. Hence *in a real three-space, if Σ_∞ is an elliptic polar system the parabolic metric geometry is the Euclidean geometry.*

The general remarks in § 66 are applicable to the three-dimensional case as well as to the two-dimensional one.

It was stated in § 66 that the congruence assumptions are no longer strictly independent when a full continuity assumption is added, because by introducing ideal elements and an arbitrary Σ_∞ (as in the present chapter) a relation of congruence may be defined for which the statements in X–XVI are theorems which can easily be proved. This view is not accepted by certain well-known mathematicians, who hold that the arbitrariness in the definition of the absolute involution somehow conceals a new assumption.* It may, therefore, be well to restate the matter here.†

Assumptions I–IX, XVII are categorical for the Euclidean space; i.e. if two sets of objects [*P*] and [*Q*] satisfy the conditions laid down for points in the assumptions, there is a one-to-one reciprocal correspondence between [*P*] and [*Q*] such that the subsets called lines of [*P*] correspond to the subsets called lines of [*Q*]. Thus the internal structure of a Euclidean space is fully determined by Assumptions I–IX, XVII. The group leaving invariant the relations described in these assumptions is the affine group, and all the theorems of the affine geometry are consequences of these assumptions. The latter may therefore be characterized as the assumptions of affine geometry.

Among the theorems of the affine geometry is one which states that there is an infinity of subgroups, each one conjugate to all the rest and such that the set of theorems belonging to it constitutes the Euclidean geometry. Each of these groups is capable of being called the Euclidean group, and there is no theorem about one of them which is not true about all of them. The set of theorems stating relations invariant under any one of these groups is the Euclidean geometry. This set of theorems is the same whichever Euclidean group be selected, i.e. *the Euclidean geometry is a unique body of theorems.*

Each Euclidean group has a self-conjugate subgroup of displacements which defines a relation called congruence having the properties stated in

* Cf. the remarks on a paper by the writer in the article by Enriques, Encyclopédie des Sc. Math. III 1, § 12.

† This discussion should be read in connection with the remarks on foundations of geometry in the introduction to Vol. I and in § 13 of this volume; also in connection with the remarks on the geometry corresponding to a group, §§ 34, 39, 1

Assumptions X–XVI. Moreover, any relation which satisfies these assumptions is associated with a group of displacements which is self-conjugate under a Euclidean group.

Thus Assumptions X–XVI characterize the relation of congruence as completely as possible, i.e. any relation satisfying these assumptions must be that determined by one of the infinitely many groups of displacements. The set of theorems about congruence is unique and is the Euclidean geometry.

The relation between the affine geometry and the Euclidean geometry is analogous to that between the Euclidean geometry and the geometry belonging to any non-self-conjugate subgroup of a Euclidean group. Consider, for example, the subgroup obtained by leaving a particular point O invariant. A relation which is left invariant by this group may be defined as follows:

DEFINITION. A point P is *nearer* than a point Q if and only if Dist (OP) $<$ Dist (OQ). P and Q are *equally near* if Dist (OP) = Dist (OQ).

There is an element of arbitrary choice in this definition, just as there is in the choice of an absolute involution to define the notion of congruence. Moreover, the geometry of *nearness* is just as truly a geometry as is the Euclidean geometry.* It would be easy to put down a set of assumptions (XVIII–N) in terms of *near* regarded as an undefined relation, which would state the abstract properties of this relation, just as X–XVI state the abstract properties of congruence.

Another non-self-conjugate subgroup of the Euclidean group which gives rise to an interesting geometry is the group leaving invariant a line and a plane on this line. In terms of this group the notions of *forward* and *backward* and *up* and *down* can be defined, and the geometry corresponding to this group is a set of propositions embodying the abstract theory of this set of relations.

It is a theorem of Euclidean geometry that the Euclidean group has subgroups with the properties involved in these geometries, just as it is a theorem of affine geometry that the affine group has Euclidean subgroups and a theorem of projective geometry that the projective group has affine subgroups.

Assumptions I–IX, XVII have a different rôle from X–XVI or XVIII–N, in that they determine the set of objects (points and lines, etc.) which are presupposed by all the other assumptions. The choice of these assumptions is logically arbitrary. The choice of such sets of "assumptions" as X–XVI is not arbitrary; it must correspond to a properly chosen group of permutations of the objects determined by I–IX, XVII. When independence proofs are given for Assumptions X–XVI, it is done by giving new interpretations to the term "congruence," not to "point" or "line."

* It is even possible to give a psychological significance to this geometry. The normal individual has a certain place, say home, in terms of nearness to which other places are thought of; here O is the central point of home. In astronomy stars are regarded as near or the contrary, according to their distance from the

The point of view of the writer is that if X–XVI or XVIII–N are to be regarded as independent assumptions, their independence is of a lower grade than that of I–IX, XVII. They constitute a definition by postulates of a relation (congruence or nearness) among objects (points, lines, etc.) already fully determined. Their significance is that they characterize that subset of the theorems deducible from I–IX, XVII which corresponds to any Euclidean group and which therefore is the Euclidean geometry.

EXERCISES

* **1.** Develop the geometry corresponding to some non-self-conjugate subgroup of the Euclidean group. Determine a set of mutually independent assumptions characterizing this geometry.

2. The identity is the only transformation of the Euclidean group which leaves fixed two points A and B and two rays (cf. definition in § 16) AC and AD orthogonal to each other and to the line AB.

3. If a and b are any two rays having a common origin, O, and on different lines, there is a unique orthogonal line reflection and a unique orthogonal plane reflection transforming a into b.

4. If A, B, C, D are any four points no three of which are collinear, there exists a unique rotation leaving the line AB invariant and transforming C into a point of the plane ABD on the same side of AB with D.

5. Any transformation of the Euclidean group which leaves a line pointwise invariant and preserves sense is a rotation.

6. Any transformation of the Euclidean group which leaves a line pointwise invariant and alters sense is an orthogonal reflection in a plane containing this line.

7. There is one and only one displacement which transforms three mutually orthogonal rays OA, OB, OC into three mutually orthogonal rays $O'A'$, $O'B'$, $O'C'$, provided that $S(OABC) = S(O'A'B'C')$.

***117. Generalization to *n* dimensions.** The discussion of the Euclidean and affine geometries in §§ 111–116 is so arranged that it will generalize at once to any number of dimensions. It is recommended to the reader to carry out this generalization in detail, at least in the four-dimensional case.

The elementary theorems of alignment for four dimensions are given in § 12, Vol. I. The definition of a Euclidean four-space is given in § 28, Vol. II. The generalization of § 111 is obvious on

A three-dimensional polar system may be defined as the polar system of a proper or improper regulus (Chap. XI, Vol. I; cf. also §§ 100–108, Vol. II), or it may be studied *ab initio* by generalizing Chap. X, Vol. I. The notion of perpendicular lines, planes, and three-spaces then follows at once and also the theorems generalizing those of § 113. An orthogonal reflection in an S_3 is next defined as a projective collineation of period two, leaving invariant a point P at infinity and each point of a three-space whose plane at infinity is polar to P in the absolute polar system. All the theorems of §§ 114, 115 up to Theorem 13 then generalize at once. Theorems 13–15 must be modified, in view of the fact that there are more than one type of four-dimensional displacements leaving a point invariant. Theorem 16 holds unchanged.

Finally, it can be proved as in § 116 that in case of a real space and an elliptic polar system the parabolic metric geometry satisfies a set of axioms for Euclidean geometry of four dimensions. This set differs from the one used above, in that VIII is replaced by

VIII′. *If A, B, C, D are four noncoplanar points, there exists a point E not in the same S_3 with A, B, C, D, and such that every point is in the same S_4 with A, B, C, D, E.*

The introduction of nonhomogeneous coördinates in a space of n dimensions may be made by direct generalizations of § 69, Vol. I. The formulas for the affine group, the group of translations, the Euclidean group, and the group of displacements are then easily seen to be identical with those given in the sections below, except that the summations from 0 or 1 to 3 must in each case be replaced by summations from 0 or 1 to n.

118. Equations of the affine and Euclidean groups. With respect to a nonhomogeneous coördinate system in which π_∞ is the singular plane, the affine group is evidently the set of all projectivities of the form

$$
\begin{aligned}
x' &= a_{11}x + a_{12}y + a_{13}z + a_{10},\\
y' &= a_{21}x + a_{22}y + a_{23}z + a_{20},\\
z' &= a_{31}x + a_{32}y + a_{33}z + a_{30},
\end{aligned}
\tag{6}
$$

where
$$
\Delta \equiv \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0,
$$

and the variables and coefficients are elements of the geometric number system.

In the system of homogeneous plane coördinates in which the plane at infinity is represented by [1, 0, 0, 0], this group takes the form

$$(7)\qquad \begin{aligned} u_0' &= b_{00}u_0 + b_{01}u_1 + b_{02}u_2 + b_{03}u_3, \\ u_1' &= \qquad\quad\; b_{11}u_1 + b_{12}u_2 + b_{13}u_3, \\ u_2' &= \qquad\quad\; b_{21}u_1 + b_{22}u_2 + b_{23}u_3, \\ u_3' &= \qquad\quad\; b_{31}u_1 + b_{32}u_2 + b_{33}u_3. \end{aligned}$$

In an ordered space the affine group has a subgroup consisting of all transformations for which Δ is positive. This group has been considered in § 31. It also has obvious subgroups consisting of all transformations for which $\Delta^2 = 1$ and for which $\Delta = 1$.

The equations of a translation parallel to the x-axis are evidently $x' = x + a$, $y' = y$, $z' = z$, and similar expressions represent a translation parallel to any other axis. Hence by the corollary of Theorem 3 the equations of the group of translations are

$$(8)\qquad \begin{aligned} x' &= x + a, \\ y' &= y + b, \\ z' &= z + c. \end{aligned}$$

If the coördinates are so chosen that the planes $\bar{x} = 0$, $\bar{y} = 0$, $\bar{z} = 0$ are mutually orthogonal, the equations of the circle at infinity in terms of the corresponding homogeneous coördinates are

$$a\bar{x}_1^2 + b\bar{x}_2^2 + c\bar{x}_3^2 = 0, \qquad \bar{x}_0 = 0.$$

These are reducible by the transformation

$$(9)\qquad x_0 = \bar{x}_0, \quad x_1 = \sqrt{a}\,\bar{x}_1, \quad x_2 = \sqrt{b}\,\bar{x}_2, \quad x_3 = \sqrt{c}\,\bar{x}_3$$

to

$$(10)\qquad x_1^2 + x_2^2 + x_3^2 = 0, \qquad x_0 = 0.$$

In the real geometry a, b, c are positive if the polar system is elliptic (§ 85), and the transformation (9) carries real points to real points. The formulas (9) are the only ones in the present section in which irrational expressions appear. Hence the rest of the discussion holds for any space satisfying Assumptions A, E, P, H_0. In any such space it is easily seen that (10) represents a conic whose polar system may be taken as Σ_∞, but it does not follow, as in the real case, that any improper conic can be reduced to this form. The situation here is entirely analogous to that obtaining in § 62.

the planes tangent to the circle at infinity (10) satisfy the equation

$$(11) \qquad u_1^2 + u_2^2 + u_3^2 = 0.$$

Any plane

$$(12) \qquad u_0 + u_1 x' + u_2 y' + u_3 z' = 0$$

is the transform under a collineation of the form (6) of the plane

$$(13) \qquad (u_0 + a_{10}u_1 + a_{20}u_2 + a_{30}u_3) + (a_{11}u_1 + a_{21}u_2 + a_{31}u_3)\,x + (a_{12}u_1 + a_{22}u_2 + a_{32}u_3)\,y + (a_{13}u_1 + a_{23}u_2 + a_{33}u_3)\,z = 0.$$

Hence (11) is the transform of

$$(14) \qquad (a_{11}^2 + a_{12}^2 + a_{13}^2)\,u_1^2 + (a_{21}^2 + a_{22}^2 + a_{23}^2)\,u_2^2 + (a_{31}^2 + a_{32}^2 + a_{33}^2)\,u_3^2 + 2\,(a_{11}a_{21} + a_{12}a_{22} + a_{13}a_{23})\,u_1u_2 + 2\,(a_{11}a_{31} + a_{12}a_{32} + a_{13}a_{33})\,u_1u_3 + 2\,(a_{21}a_{31} + a_{22}a_{32} + a_{23}a_{33})\,u_2u_3 = 0.$$

In order that (11) and (14) shall represent the same locus, we must have

$$(15) \qquad \begin{aligned} &a_{11}^2 + a_{12}^2 + a_{13}^2 = a_{21}^2 + a_{22}^2 + a_{23}^2 = a_{31}^2 + a_{32}^2 + a_{33}^2, \\ &a_{11}a_{21} + a_{12}a_{22} + a_{13}a_{23} = a_{11}a_{31} + a_{12}a_{32} + a_{13}a_{33} \\ &\qquad = a_{21}a_{31} + a_{22}a_{32} + a_{23}a_{33} = 0. \end{aligned}$$

These conditions are equivalent to the equation (cf. § 95, Chap. X, Vol. I)

$$(16) \qquad \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \cdot \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix} = \begin{pmatrix} \rho & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & \rho \end{pmatrix},$$

where $\rho = a_{11}^2 + a_{12}^2 + a_{13}^2$.

If the matrix $(a_{11}a_{22}a_{33}) = A$ be interpreted as the matrix of a planar collineation, as in § 95, Vol. I, this states that the product of the collineation by the collineation represented by the transposed matrix is the identity. Hence the product of the two matrices in the reverse order is a matrix representing the identity. This means that

$$a_{11}^2 + a_{21}^2 + a_{31}^2 = a_{12}^2 + a_{22}^2 + a_{32}^2 = a_{13}^2 + a_{23}^2 + a_{33}^2,$$

and

$$a_{11}a_{12} + a_{21}a_{22} + a_{31}a_{32} = a_{11}a_{13} + a_{21}a_{23} + a_{31}a_{33} = a_{12}a_{13} + a_{22}a_{23} + a_{32}a_{33} = 0.$$

Since the determinants of a matrix and of its transposed matrix are equal, we have

$$\Delta^2 = \rho^3 = (a_{11}^2 + a_{12}^2 + a_{13}^2)^3 = (a_{11}^2 + a_{21}^2 + a_{31}^2)^3.$$

DEFINITION. A matrix such that its product by a given matrix A is the identical matrix (§ 95, Vol. I) is called the *inverse* of A and is denoted by A^{-1}. A square matrix whose transposed matrix is equal to its inverse is called *orthogonal.* A linear transformation,

$$(17)\qquad \begin{aligned} x' &= a_{11}x + a_{12}y + a_{13}z, \\ y' &= a_{21}x + a_{22}y + a_{23}z, \\ z' &= a_{31}x + a_{32}y + a_{33}z, \end{aligned}$$

whose matrix $(a_{11}a_{22}a_{33})$ is orthogonal, is said to be *orthogonal.*

The results at which we have arrived may now be expressed in part as follows:

THEOREM 17. *The transformations of the parabolic metric group can be written in the form*

$$(18)\qquad \begin{aligned} x' &= \rho\,(a_{11}x + a_{12}y + a_{13}z + k_1), \\ y' &= \rho\,(a_{21}x + a_{22}y + a_{23}z + k_2), \\ z' &= \rho\,(a_{31}x + a_{32}y + a_{33}z + k_3), \end{aligned}$$

where the matrix $(a_{11}a_{22}a_{33})$ *is orthogonal.*

From the form of these equations we obtain the following corollaries:

COROLLARY 1. *Any transformation* (18) *of the Euclidean group is the product of an orthogonal transformation, a translation, and a homology of the form*

$$(19)\qquad \begin{aligned} x' &= \rho x, \\ y' &= \rho y, \\ z' &= \rho z. \end{aligned}$$

COROLLARY 2. *A homology* (19) *is commutative with any collineation leaving the origin invariant.*

Since an orthogonal matrix is any matrix satisfying (16) with $\rho = 1$, we have

COROLLARY 3. *The product of two orthogonal transformations is orthogonal. The determinant of an orthogonal transformation is* $+1$ *or* -1.